Physics 161: Black Holes: Lecture 3: 8 Jan 2010

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3 Metrics Continued

It was mathematicians who first wondered whether a consistent system could be created with different (non-Euclidean) metrics. Gauss, Bolyai, Lobachevski, Reimann, etc. worked it all out. When Einstein came along with his great intution that this might be relevant to the real world he applied this math, "differential geometry" in his GR.

GR takes the matter and energy in a system and predicts the metric. This is done by solving Einstein's field equations, something we will not do in this course. Once you have the metric you can calculate distances, times, and motions of particles. We will do this. GR gives some surprising results, and so far every prediction that has been tested has been experimentally verified.

3.1 Expanding Universe and FRW metric

For example, consider the entire Universe filled uniformly with matter. One can solve the field equations for the metric in this case and find (for the right amount of matter):

$$ds = R(t)\sqrt{dx^2 + dy^2 + dz^2},$$

where the *scale factor*, $R(r) = (t/t_0)^{1/2}$ at early times and $R(t) = (t/t_0)^{2/3}$ or $R(t) = \exp(Ht/t_0)$ at late times depending on the cosmology. Here $t_0 = 13.7$ Gyear is the age of the Universe today. This is called the Friedmann, Robertson, Walker (FRW) metric and was quite shocking to Einstein when he first realized this is what his GR theory predicted for uniform matter. The shocking thing is that $R(t)$ changes with time. Thus the distance between objects changes with time, even when they are "at rest"! Consider two galaxies a distance D apart today (one at (x_0, y_0, z_0) and the other at (x, y, z). Distance between them is $s = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, with $t = t_0$. A few years from now they are farther apart by a factor $(t/t_0)^{2/3}$ even though their positions, x, x_0, y, y_0 etc. haven't changed!

This is the expansion of the Universe. Note it is *not an explosion* that happened long ago and caused everything to blast apart. Note also what happens when $t = 0$ in this metric. $R(0) = 0$, implying $s = 0$ no matter what the values of x, y, etc. are. That is, at $t = 0$, everything in the Universe is touching everything else! This is the big bang. This metric predicts the big bang happened at $t = 0$, and that the Universe expanded since then because the metric is changing with time.

3.2 Spacetime metrics and nomenclature

Spacetime metrics combine the 3-D space metric with the time metric. The flat spacetime metric, also known as the Minkowski metric, is written in Cartesian coordinates as:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

This combines the flat space 3-D metric with the flat time metric: c^2dt^2. The two are combined together so that ds is a Lorentz invariant, that is if the time between two events is dt, and the distance between those same two events is $\sqrt{dx^2 + dy^2 + dz^2}$, then the spacetime interval between these two events has the same value (ds), *no matter which inertial reference frame is used to make the measurement.* It is crucial for this to work that the "time part" of the metric have a different sign that the "space part".

One uses the spacetime metric to meausure the "invariant spacetime interval", ds, between two events. One also uses the space part of the metric to measure real distances (distances measured by a meter stick). The time part of the metric is used to measure real (that is measured by a clock) times between events. The distance between two events is best measuared when the time of the events is the same, that is measured at the same time, or by setting $dt = 0$. Thus the **proper distance** between two objects is defined as $dl = \sqrt{ds^2}$, with $dt = 0$. To measure times between events, the events should be at the same position, i.e. one clock at different times. So we set $dx = dy = dz = 0$. The **proper time** between events is defined as $d\tau = \sqrt{-ds^2}/c$, Note, that because the time part of the metric has a minus sign, you have to add a minus sign to cancel it out. While ds^2 can be positive, negative, or zero, the invariant interval, ds is always positive. You should never get an imaginary number out of a metric!

Actually the sign of ds^2 is very important:

- $ds^2 < 0$ **implies a time-like interval**, meaning the two events can be causally related, that is, it is possible for the event with the earlier time to have caused the event with the later time.
- $ds^2 > 0$ **implies a space-like interval.** It is impossible for either of these events to have caused the other. (Like the ends of the same ruler.)
- $ds^2 = 0$ **implies a null or light-like interval.** These two events can be connected by light rays only. Note in this case the definitions of proper time and proper distances above don't apply because you can't set $dt = 0$ or $dx = 0$. The proper time between lightlike separated events is defined as $ds = 0$.

Note that with the Minkowski metric, the variable t measures time and the variable x measures spatial distance. However, in more general metrics, the variables used in the metric *do not represent time or space directly.* To get actual time or space measurements you have to use the left-hand side of the metric, i.e. ds, as described above. The variables in the metric are called the "coordinate time" or "coordinate distances", or even just the coordinates, and may or may not correspond to clock time, or meter-stick distance.

Also, note that many books (and I myself) set the speed of light, c, equal to one. This makes the units easier to work throughout all of Special and General Relativity. It means if you are measuring time in years, you are measuring distance in light-years, or if you are measuring distance in meters, you are measuring time in units of the time for light to travel one meter. A useful thing to know is that light travels about 1 foot in one nanosecond. When using these units, you have to add back in powers of c in order to get to useful units. There is *always* a unique way to do this. In these units velocities are dimensionless, so if you want a velocity in m/s just multiple by $c = 3 \times 10^8$ m/s. Energies and masses are related by $E = mc^2$, so if you know the mass and want an energy just multiply by c^2. [e.g. $v = .001$ means, $v = .001c$, or $v = 300$km/s.]

Finally note that in many books the space part of the metric has the minus signs, and the time part is positive. As long as there is a relative minus sign between space and time, it doesn't matter which has the minus sign, but it is very important to pick one convention and keep it. This is called the "signature of the metric". We will use the "East coast" signature (- + + +), while others use the "West coast" signature (+ - - -). With the opposite signature metric, the definitions of proper distance and proper time change: e.g. $d\tau = \sqrt{ds^2}/c$, $dl = \sqrt{-ds^2}$, and several other formulas change as well.

3.3 Schwarzschild metric

The FRW metric is valid on very large scales where matter is distributed approximately uniformly. On scales the size of the Earth, Solar System, or even galaxies, matter is concentrated in a central source, so FRW is not the proper solution of Einstein's field equation and is not the correct metric. The simplest case is for a spherical object, such as the Earth, Sun, (or black hole!). Solving Einstein's field equations for the region outside of a spherical object of mass, M gives the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

where this is in spherical coordinates with $r^2 = x^2 + y^2 + z^2$. There are several new features here and we will spend a lot of time on this metric.

- First note that flat 3-D spatial metric in spherical coordinates is $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 = dr^2 + r^2 d\Omega^2$, **Note $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is shorthand for the angular part.** This looks similar to, but is different that the 2-D curved metric on the surface of the sphere because here r is a variable not a constant. You can tell the variables because they are differential (i.e. dr not only r).
- Second, I included the time part of the metric dt. This makes it a spacetime metric, and not just space. We will talk more about this.
- Third, the time part of the metric has a minus sign! We will come back to what that means and how to deal with it.
- Fourth, the metric does not change with time. Unlike the FRW metric, the variable t does not appear explicitly anywhere in the Schwarzschild metric. The dt doesn't count since that is just tells how to measure time. Thus this metric is not expanding or contracting and just sits there. (Good thing).
- Fifth, the metric is spherically symmetric, which is why we switched to spherical coordinates.
- Sixth, there are factors in front of both dr and dt which mean that both the space and the time are curved in this metric. If those factors were equal to 1, then this would be the 3-D flat spacetime Minkowski metric.
- Seventh, note that something weird happens when $2GM/rc^2 = 1$. The dt^2 term goes to zero and the dr^2 term goes to infinity. This happens when a mass M is squeezed into a ball of radius

$$r_{SC} = \frac{2GM}{c^2},$$

where r_{SC} is called the Schwarzschild radius. This is the event horizon radius of a black hole, and we expect some weird things to happen at that radius. We will discover that whenever the size of an object is smaller than its Schwarzschild radius it is a black hole. For the mass of the Sun this radius is

$$r_{SC} = \frac{(2)(6.67 \times 10^{-11}\text{m}^3/\text{kg s}^2)(2 \times 10^{30}\text{kg } M/M_\odot)}{(3 \times 10^8\text{m/s})^2} = 3\text{km } \frac{M}{M_\odot},$$

where $M_\odot$ is the mass of the Sun. Thus if you could jam the entire mass of the Sun into a sphere of radius 3 km, it would be a black hole.

- Finally, note that if the key factor $2GM/rc^2$ is small, then the Schwarzschild metric is very close to the 3-D flat space metric. For the Sun, $2GM/rc^2 = 3$ km$/7 \times 10^5$km $= 4 \times 10^{-6}$. So even at the surface of the Sun, spacetime is only 4 parts in a million away from being flat. The smallness of this number is one reason why it is hard to measure the effects of GR that differ from Newton's law. Around the Earth spacetime is even closer to flat. Note that far away from the mass, $r \to \infty$, the Schwarzschild metric turns into exactly the flat space metric. This is what we expect of course; there is no gravity infinitely far away from a mass.