Physics 211B : Problem Set #2

[**1**] Consider a disordered tight binding model, for which the Schrödinger equation is

$$-t\,\psi_{n-1} - t\,\psi_{n+1} + \varepsilon_n\,\psi_n = E\,\psi_n \ ,$$

where t is the hopping integral and ε_n is the local site energy. Compute the Landauer resistance $\mathcal{R} = |r|^2/|t|^2$ for a disordered region of length N as a function of the incident wavevector k. Your expression should involve a product of transfer matrices for the individual sites. Show that $\langle \mathcal{R}(N) \rangle \propto e^{N/\zeta}$, where ζ is determined by an eigenvalue problem. *Hint: Show that $\mathcal{R}$ is related to Tr$(\mathcal{M}^\dagger \mathcal{M})$, where $\mathcal{M}$ is the full transfer matrix relating in and out plane wave states on either side of the disordered region.*

Suppose that ε_n on each site is uniformly distributed on the interval $\big[-\frac{1}{2}W, \frac{1}{2}W\big]$. Then $\zeta = \zeta(E/t, W/t)$. Plot ζ *versus* E/t for several representative values of W/t.

[**2**] Derive eqn. 2.76 in the notes:

$$\Big[\mathcal{M}^\dagger \mathcal{M} + (\mathcal{M}^\dagger \mathcal{M})^{-1} + 2 \cdot \mathbb{I}\Big]^{-1} = \frac{1}{4} \begin{pmatrix} t^\dagger t & 0 \\ 0 & t' t'^\dagger \end{pmatrix} \ ,$$

were $\mathcal{M}$ is the transfer matrix for a multichannel system.

[**3**] Graphene is a two-dimensional honeycomb lattice of carbon atoms. The $2s$ and $2p$ electrons undergo planar sp^2 hybridization. The remaining p_z orbital comprises the π-band. It is half-filled, since there are a total of four electrons in the $2s$ and $2p$ orbitals for each carbon atom.

(a) Solve for the tight-binding band structure of the π-band, assuming nearest neighbor hopping with amplitude t on a honeycomb lattice. Remember that a honeycomb lattice may be represented as a triangular Bravais lattice with a two-element basis.

(b) Show that near the two inequivalent Brillouin zone corners the spectrum is Dirac-like, *i.e.*

$$H \approx -i\hbar v_{\rm F} \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \ ,$$

where $v_{\rm F}$ is the slope of the Dirac cone, and where $\boldsymbol{\sigma}$ are the Pauli matrices which act on the sublattice index, which may be considered a pseudospin. Find $v_{\rm F}$ in terms of t, the lattice constant a, and other constants.

(c) For the Dirac Hamiltonian above, compute the current density. *Hint: recall* $\boldsymbol{p} \to \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}$.

(d) Now consider a two-dimensional Dirac particle propagating in a piecewise constant potential $V(x)$ which is a function of x alone and not y. Show that for every direction of propagation θ there are two eigenstates

$$\psi_\pm(x,y) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm\, e^{i\theta} \end{pmatrix} e^{ik(x\cos\theta + y\sin\theta)} \ ,$$

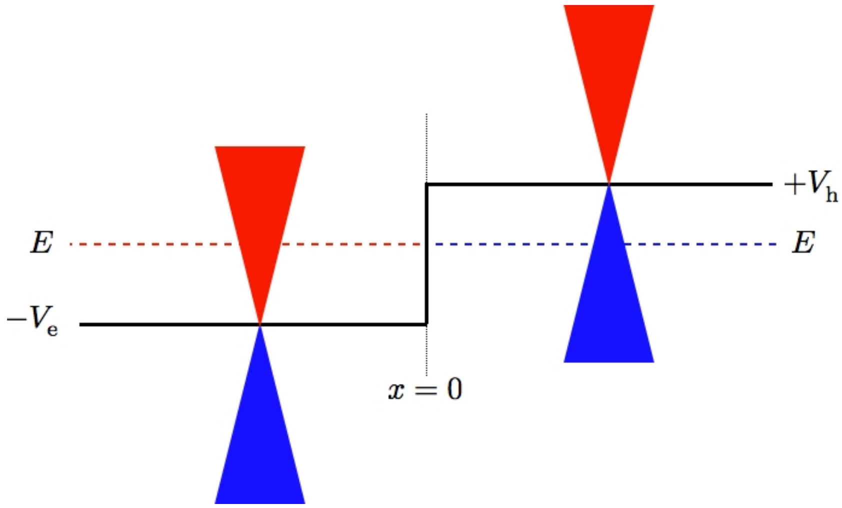

Figure 1: A potential step from $V(x<0) = -V_\text{e}$ to $V(x>0) = +V_\text{h}$.

with $E_\pm = V \pm \hbar v_\text{F} k$, where V is the local value of $V(x)$. Without loss of generality we may take $k \geq 0$.

(e) Consider next a p-n junction modeled as a step potential, as depicted in fig. 1. Show that continuity of the wavefunction across the boundary requires that the angles θ_e and θ_h are related by Snell's law. Under what condition does an electron wave propagating from the left ($\cos\theta_\text{e} > 0$) undergo total internal reflection?

(f) Find the transfer matrix M relating the amplitude of the rightming and leftgoing propagating wave solutions across the interface, writing

$$x<0\ :\ \ \psi_<(x,y) = \left[\frac{A_1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\theta_\text{e}}\end{pmatrix} e^{ik_\text{e}\,x\cos\theta_\text{e}} + \frac{A_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{-i\theta_\text{e}}\end{pmatrix} e^{-ik_\text{e}\,x\cos\theta_\text{e}}\right] e^{ik_\text{e}\,y\sin\theta_\text{e}}$$

$$x>0\ :\ \ \psi_>(x,y) = \left[\frac{B_1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\theta_\text{h}}\end{pmatrix} e^{ik_\text{h}\,x\cos\theta_\text{h}} + \frac{B_2}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\theta_\text{h}}\end{pmatrix} e^{-ik_\text{h}\,x\cos\theta_\text{h}}\right] e^{ik_\text{h}\,y\sin\theta_\text{h}} \ .$$

[3] Consider the ring geometry in fig. 2. There is a single scatterer, with $V(x) = \Omega\,\delta(x)$, located at the position shown by the star. Each T junction is described by the S-matrix

$$S = \begin{pmatrix} -(a+b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & a & b \\ \sqrt{\epsilon} & b & a \end{pmatrix} \ ,$$

with $\mathsf{S}^\dagger\mathsf{S} = \mathbb{I}$ and $\mathsf{S}(\epsilon = 0) = \text{diag}(-1,+1,+1)$. Compute the dimensionless two-terminal conductance, $g = hG/e^2$, as a function of kR, QR, α, γ, and ϵ, where R is the ring radius, $Q = m^*\Omega/\hbar^2$, and $\gamma = e\Phi/\hbar c$. The dispersion in the leads is $E(k) = \hbar^2k^2/2m^*$.

You likely won't obtain an expression in closed form. Write a computer program to compute $g(kR, QR, \alpha, \gamma, \epsilon)$, and make some plots of g *versus* kR and g *versus* γ for different values

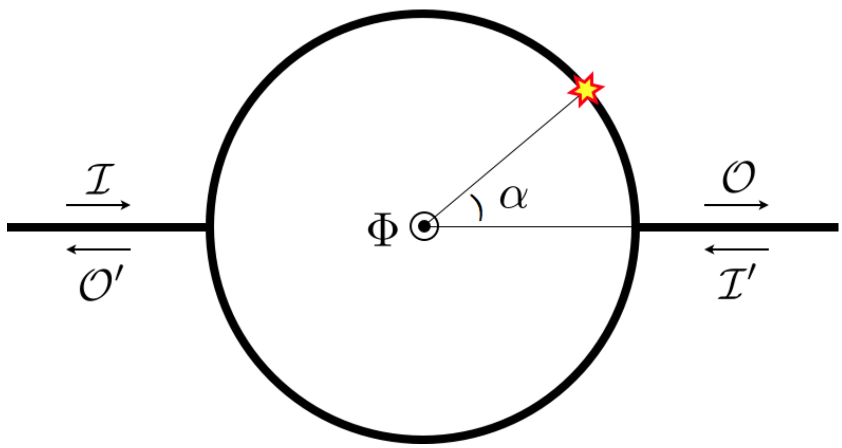

Figure 2: A mesoscopic ring enclosing with a single scatterer enclosing magnetic flux Φ.

of the parameters QR, α, and ϵ. Explore how the curves change when the other parameters are varied.