Physics 211B : Final Exam

Due 10 am, Wednesday March 17, my office (5671 Mayer Hall)

[1] Consider a tri-junction connecting three identical one-dimensional leads. The leads are each connected to reservoirs described by chemical potentials μ_α at a fixed temperature T. The $\mathcal{S}$-matrix relates incoming and outgoing plane wave states, as usual:

$$\begin{pmatrix} A^{\text{OUT}} \\ B^{\text{OUT}} \\ C^{\text{OUT}} \end{pmatrix} = \begin{pmatrix} r_A & t_{AB} & t_{AC} \\ t_{BA} & r_B & t_{BC} \\ t_{CA} & t_{CB} & r_C \end{pmatrix} \begin{pmatrix} A^{\text{IN}} \\ B^{\text{IN}} \\ C^{\text{IN}} \end{pmatrix} .$$

(a) Following the arguments in §2.3 of the lecture notes, derive, *mutatis mutandis*, an equation relating the current I_α in terms of the chemical potentials μ_α of the reservoirs. Be sure to comment on aspects such as current conservation.

Now consider the tight binding tri-junction model described in fig. 1. The hopping matrix elements along the chains are all identical and are equal to t. The hopping matrix elements on the internal triangle are all identical and equal to $t_\triangle$. The on-site energies for all sites are identical and equal to $\varepsilon_0 = 0$.

(b) Derive an expression for the $\mathcal{S}$-matrix for this system. You should write

$$A_n = A^{\text{IN}}\, e^{-ikn} + A^{\text{OUT}}\, e^{+ikn} ,$$

with corresponding expressions for the B and C leads. (See the hint at the end of the problem for some mathematical guidance.)

(c) Suppose $\mu_A = eV$ and $\mu_B = \mu_C = 0$. Derive an expression for the current I_B at $T = 0$. Plot the dimensionless conductance $(h/e^2) \times (I_B/V)$ *versus* the dimensionless incident energy $\varepsilon = E/t$ over the allowed range $\varepsilon \in [-2\,,\,2]$ for several values of the ratio $r \equiv t_\triangle/t$.

Hint: At some point, you may find it necessary to invert a matrix of the form

$$R = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} .$$

To this end, note that we can write

$$R = (a-b)\,\mathbb{I} + 3b\,|\,\psi\,\rangle\langle\,\psi\,| ,$$

where $\vec{\psi}^{\text{T}} = \frac{1}{\sqrt{3}}\,(1\,,\,1\,,\,1)$, so $|\,\psi\,\rangle\langle\,\psi\,|$ is a matrix whose elements are all equal to $\frac{1}{3}$. But then

$$R = (a-b)\,Q_\psi + (a+2b)\,P_\psi ,$$

where $P_\psi = |\,\psi\,\rangle\langle\,\psi\,|$ is the projector onto $|\,\psi\,\rangle$, and $Q_\psi = \mathbb{I} - P_\psi$ is the projector onto the two-dimensional subspace orthogonal to $|\,\psi\,\rangle$. But then, clearly

$$R^{-1} = \frac{1}{a-b}\,Q_\psi + \frac{1}{a+2b}\,P_\psi = \frac{1}{(a-b)(a+2b)} \begin{pmatrix} a+b & -b & -b \\ -b & a+b & -b \\ -b & -b & a+b \end{pmatrix} .$$

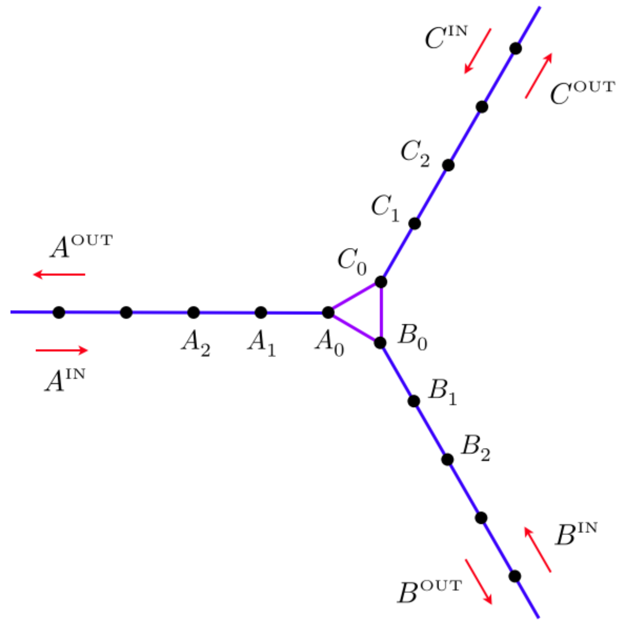

Figure 1: A tri-junction formed from three semi-infinite single-orbital tight-binding chains.

[2] Consider a spin-S quantum Heisenberg model on a bipartite lattice. The A sublattice sites are located at positions $\boldsymbol{R}$ and the B sublattice sites at $\boldsymbol{R}+\boldsymbol{\delta}$, where $\boldsymbol{R}$ is an element of some Bravais lattice and $\boldsymbol{\delta}$ is the sole basis vector. The Hamiltonian is

$$\mathcal{H} = -\sum_{\boldsymbol{R},\boldsymbol{R}'} \Big\{ \tfrac{1}{2}\, J_{\rm AA}\big(|\boldsymbol{R}-\boldsymbol{R}'|\big)\, \boldsymbol{S}_{\rm A}(\boldsymbol{R})\cdot \boldsymbol{S}_{\rm A}(\boldsymbol{R}') + \tfrac{1}{2}\, J_{\rm BB}\big(|\boldsymbol{R}-\boldsymbol{R}'|\big)\, \boldsymbol{S}_{\rm B}(\boldsymbol{R})\cdot \boldsymbol{S}_{\rm B}(\boldsymbol{R}') + J_{\rm AB}\big(|\boldsymbol{R}-\boldsymbol{R}'-\boldsymbol{\delta}|\big)\boldsymbol{S}_{\rm A}(\boldsymbol{R})\cdot \boldsymbol{S}_{\rm B}(\boldsymbol{R}')\Big\} - \gamma \sum_{\boldsymbol{R}} \Big\{ H_{\rm A}(\boldsymbol{R})\, S^z_{\rm A}(\boldsymbol{R}) + H_{\rm B}(\boldsymbol{R})\, S^z_{\rm B}(\boldsymbol{R})\Big\}$$

where $\boldsymbol{S}_{\rm A}(\boldsymbol{R})$ is the spin operator at the A sublattice site located at $\boldsymbol{R}$, and $\boldsymbol{S}_{\rm B}(\boldsymbol{R})$ is the spin operator at the B sublattice site located at $\boldsymbol{R}+\boldsymbol{\delta}$.

(a) Compute the susceptibility

$$\chi_{\rm AB}(\boldsymbol{q}) = \frac{\partial M_{\rm A}(\boldsymbol{q})}{\partial H_{\rm B}(\boldsymbol{q})}\bigg|_{H_{\rm A}=H_{\rm B}=0}$$

using a mean field approach. Recall the local susceptibility for a single Heisenberg spin is $\chi_0(T) = \gamma^2 p^2/k_{\rm B}T$, where $p^2 = \frac{1}{3}S(S+1)$. (You should express your answer in terms of χ_0 and other relevant quantities.)

(b) Consider the model on a honeycomb lattice. The AB interactions are between nearest neighbors only, and are given by $J_{\rm NN} < 0$ (antiferromagnetic). The AA interactions are between next-nearest neighbors only, and are given by $J_{\rm NNN} > 0$ (ferromagnetic). Find an expression for $T_{\rm c}$.

Big hint: You should derive an equation of the form $R_{ab}(\boldsymbol{q})\, M_b(\boldsymbol{q}) = H_a(\boldsymbol{q})$, where a and b run over sublattices and $R(\boldsymbol{q})$ is some matrix. The susceptibility matrix is the inverse of $R(\boldsymbol{q})$, and $\chi_{\rm AB}(\boldsymbol{q})$ is the upper right element. To find $T_{\rm c}$, set $\det(R) = 0$.

(c) Consider a nearest-neighbor Heisenberg antiferromagnet on the honeycomb lattice with an easy axis anisotropy term. The Hamiltonian is

$$\mathcal{H} = J \sum_{\langle ij \rangle} \Big(S_i^x\, S_j^x + S_i^y\, S_j^y + \Delta\, S_i^z\, S_j^z \Big) \ ,$$

where $J > 0$ and $\Delta > 1$. Derive the spin wave spectrum. For 10^{50} quatloos extra credit, plot the spin wave dispersion on a triangle Γ–K–M–Γ in the Brillouin zone.