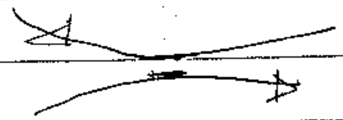
Simple Ideas in Non-Ideal MHD

→ Freezing-in law:

$$\frac{d\underline{B}}{dt} = \underline{B}\cdot\nabla\underline{v} + \eta\nabla^2\underline{B}$$

breaking ⇒ small scale → singularity / turbulence

→



→ current sheet
singular layers

→ sites of reconnection → boundary layer problem.

→ topology change

→ So

1) – Sweet-Parker Reconnection theory

3) – Re-visiting magnetic helicity; Taylor Theory

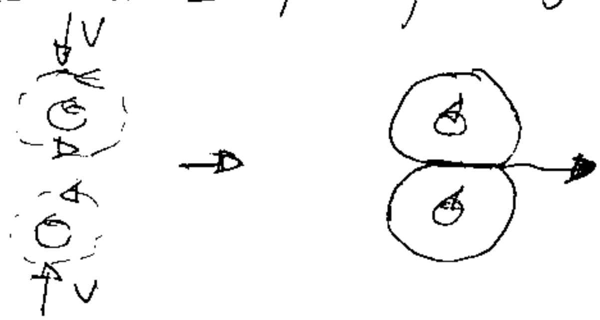
4) – anomalous resistivity (again) – later

2) – flux expulsion

→ Breakdown of Flux Freezing – Magnetic Reconnection ?!

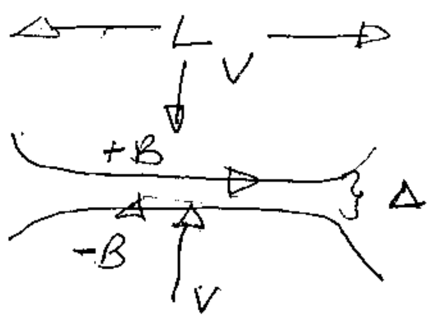
Simple Example : Sweet – Parker Problem
(re-visit later)

→ consider two cylinders of plasma, carrying current ⊥ plane, brought together (could arise in turbulence)



→ reconnection layer (Δ small → η/Δ^2 significant)
↓
flux freezing broken

⇒ consider layer



current sheet = singularity

$\Delta < L$

2 plasma slabs brought together at V

{ What Happens ?
Stationary Solution Possible ? }

$\nabla \cdot \underline{V} = 0$

$$\frac{d}{dt}\underline{B} = \underline{B}\cdot\underline{\nabla V} + \eta \nabla^2 \underline{B}$$

$$S_{ij}^{(0)} = \begin{pmatrix} 0, & 0 \\ 0, & -V\delta(y) \end{pmatrix}$$

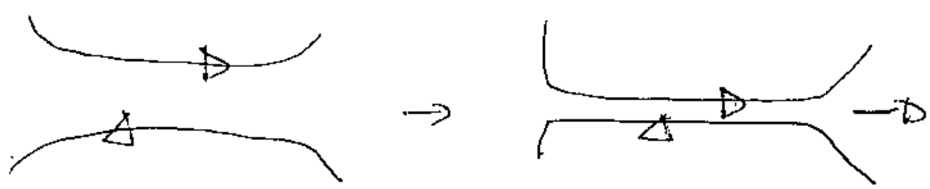
rate-of-strain tensor

→ singularity !

tip-off of small scale generation in $\underline{B}$
⇒ resistive diffusion, breaking of freezing in . .

i.e. for stationary solution,

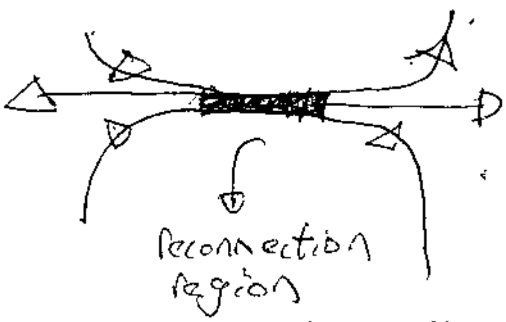
$$-\frac{\underline{B}\cdot\underline{\nabla}\,\underline{V}}{\eta} = \nabla^2\underline{B}\,, \qquad \underline{\nabla}\cdot\underline{V} = 0$$

⇒ 

→ Δ small

$\nabla(B^2/8\pi)$ increasing

fluid expelled ,

⇒ 

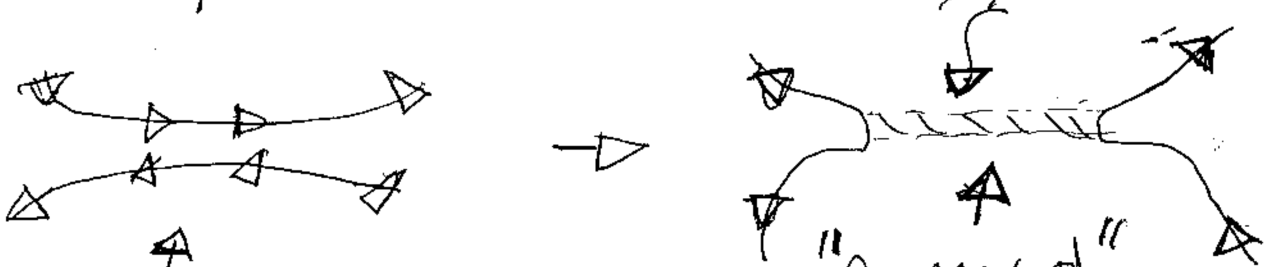
out flow along B_o

expect opposed fields
"annihilate"

(large: resistive dissipation)

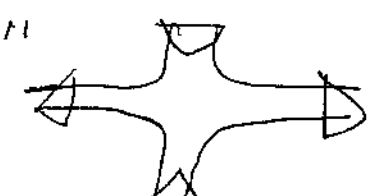
N.B. A particular V value is required for stationarity

N.B. → why "reconnection"?



initial magnetic configuration

"reconnected" configuration

→ Flow is "stagnation point" 

→ How Calculate ? → Match In-Flow → Out-Flow
(S-P. is a great Back-of-Envelope ...)

Conserved : ① − mass $(\underline{\nabla}\cdot\underline{V} = 0)$

② − momentum in $\hat{x}$ direction (symmetry)

③ − energy balance →
→ rate of field 'delivery' to reconnection region
MUST BALANCE
→ rate of Ohmic dissipation $E\cdot J \sim \eta J^2$

①

$$\rho_0 \underset{\text{inflow}}{V} \overset{\text{extent in } \hat{x}}{L} = \rho_0 \underset{\text{outflow}}{V_0} \overset{\text{extent in } \hat{y}}{\Delta}$$

$$VL = V_0 \Delta$$

$$V = V_{out}\, \Delta / L$$

② $$\rho_0\left(\frac{\partial V}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}B}{4\pi}$$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{V}\times\underline{\omega}$$

symmetry: $$0 = \underline{\nabla}\left(P + \frac{B^2}{8\pi} + \frac{\rho_0 V^2}{2}\right)$$

modified Bernoulli Eqn.

$a \rightarrow V = 0,\ B$ finite

$\dot{a} \quad \dot{b} \qquad b \rightarrow V = V_{out},\ B \rightarrow 0 \qquad \left(\frac{B^2}{8\pi} \ll P\right)$

so $P + \frac{B^2}{8\pi} + \frac{\rho_0 V^2}{2} = \text{const.}$

$$P + \frac{B^2}{8\pi} = P + \frac{\rho_0 V_{out}^2}{2}$$

$$V_{out}^2 = B^2/4\pi\rho_0 = V_A^2$$

↳ Alfven speed

$V_{out} = V_A$

$V = V_A \frac{\Delta}{L}$

specifies "speed V" in terms Δ.

n.b. OH heating → effects other on P → validity of uniformity

③ 

Energy balance

⇒

$$\begin{pmatrix}\text{Rate of Magnetic} \\ \text{Energy Inflow}\end{pmatrix} = \begin{pmatrix}\text{Rate of Ohmic} \\ \text{Dissipation, net}\end{pmatrix}$$

$P_{OH} = \frac{J^2}{\sigma}\Delta L$ so $\dot{E}_{OH} = \frac{J^2}{\sigma} L\Delta$

$$= \left(\frac{c}{2\pi}\right)^2 \frac{B^2}{\Delta^2}\frac{L\Delta}{\sigma}$$

$\nabla \times B = \frac{4\pi}{c} J$

$2B = \frac{4\pi}{c} J \Delta$

$P_{in} = 2\left(B^2/8\pi\right) V L = \dot{E}_{in}$

n.b.

$\eta J^2 \sim \frac{\eta B^2}{\Delta^2} \sim \eta^{(0)} B^2$

balance ⇒ $\cancel{2}\left(\frac{\cancel{B^2}}{\cancel{8\pi}}\right) V \cancel{L} = \frac{c^2}{\cancel{4}\pi\cancel{4}\sigma}\frac{\cancel{B^2}}{\Delta^2}\cancel{L}\Delta$

$$V = \left(\frac{c^2}{\pi\sigma}\right)/\Delta \sim \frac{\eta}{\Delta}$$

$$\frac{c^2}{4\pi\sigma} \equiv \eta \left(\sim \frac{L^2}{T}\right)$$

$V = V_A \, \Delta / L$

$V = \eta / \Delta$

$$\Rightarrow \quad \frac{\Delta}{L} = \left(\frac{\eta}{L V_A}\right)^{1/2} = \left(1/R_m\right)^{1/2}$$

and

$$V = V_A / \sqrt{R_m}$$

$R_m = \dfrac{VL}{\eta} \equiv$ Magnetic Reynolds # (here with $V = V_A$)

$\Rightarrow$ Punch Line: (for large R_m)

① – layer is thin $\quad \dfrac{\Delta}{L} \sim 1/\sqrt{R_m}$

– speed is faster than η/L, slower than V_A $\quad V \sim V_A/\sqrt{R_m}$

② Flow pattern is a/a' stagnation $\Rightarrow$ ejection from reconnection layer at V_A

issue: $1/\tau_R = \dfrac{V}{L} \sim 1/\tau_A\sqrt{S} \sim 1/\tau_A\sqrt{R_m}$

slow

R_m dependence $\rightarrow$ [illegible] $\rightarrow$ away from outflow

i.e. balance masses over long, thin layer

$\Rightarrow$ value L?

$\rightarrow$ symmetry

Moral of this story:

$\rightarrow$ Freezing-in violated when flows bring opposing $\underline{B}$ into contact $\quad \overrightarrow{}$, $\overleftarrow{}$

$\Rightarrow$

$\rightarrow$ generates singularities $\rightarrow$ thin current layers, which alter critical magnetic topology

$\Rightarrow$ "magnetic reconnection", "tearing", etc..

$\lim_{\eta \to 0} \neq \nu_{\to 0}$ ob chito → why field at vortices? 1.

> Flux Expulsion and Homogenization - Non-Ideal, cont'd

- so far, have encountered:

→ S-P reconnection ⇒ weak dissipation ($R_m >> 1$) has strong effect at [redacted] singularity - BOUNDARY LAYER

→ Taylor Hypothesis ⇒ small flux tubes destroyed by stochasticity, leaving $\int d^3x\, \underline{A} \cdot \underline{B}$ as robust invariant.

∴ diffusive dissipation most effective at breaking freezing-in on small scales

5:1N → Another example: { singular behavior in 2D closed-streamline flow }

→ Homogenization Theory → { Prandtl, Batchelor, Weiss → '66 Phil Trans, Rhines, Young }

recall $\underline{\omega}$ evolution for $\underline{\nabla} \cdot \underline{V} = 0$

$$\frac{\partial}{\partial t} \underline{\omega} + \underline{V} \cdot \underline{\nabla} \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} V + \nu \nabla^2 \omega$$

2D → $\underline{\omega} \cdot \nabla V = 0$ $\qquad \omega = \omega(\hat{z})$, $V = \nabla \phi \times \hat{z}$

c.f. P.A. Davidson, 2004 - c.f.

then $\partial_t \omega + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} \omega = \nu \nabla^2 \omega$

more generally scalar ξ: $\left\{ \begin{array}{l} \text{active} \\ \text{or} \\ \text{passive} \end{array} \right.$

$$\partial_t \xi + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} \xi = \nu \nabla^2 \xi$$

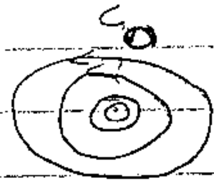
Now: $t \to \infty$, $\partial_t \xi \to 0$

$$\underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} \xi = \nu \nabla^2 \xi$$

$\nu \to 0$ $\quad \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} \xi = 0$

i.e. $Pe \sim \frac{VL}{\nu} \to \infty$ $\quad \xi = \xi(\phi)$
$\sim Re$

i.e. bounded domain, closed streamline solution

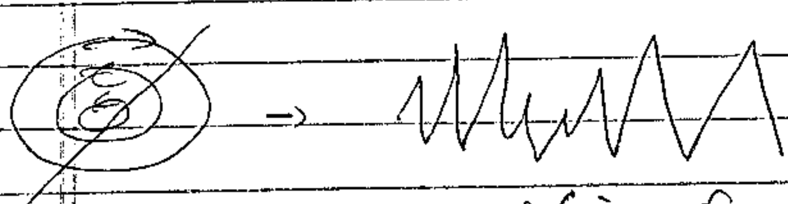


$\to \xi = \xi(\phi(r))$ is arbitrary solution

can develop arbitrarily fine scale $\xi(\phi)$ → closed streamlines ⇒ perfect memory

→ ∴ fine scale structure develops, no inter-streamline communication, & persists

i.e.



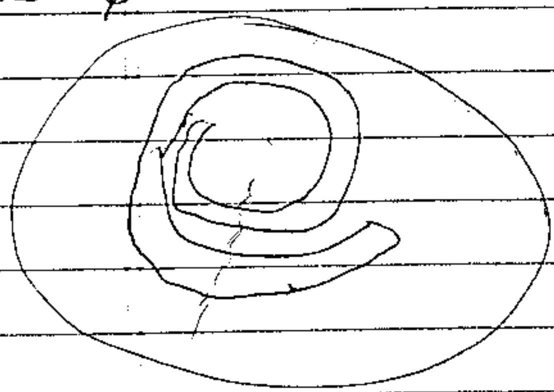
$\left\{ \begin{array}{l} \text{i.e. tag each} \\ \text{streamline} \\ \text{arbitrarily} \end{array} \right.$

~ no smoothing of sharp gradients

"Not all solutions of the Navier-Stokes equations are realized in nature."
Landau & Lifshitz

→ i.e. of  → blob in concentric shear flow

blow-up



⇒ non-diffusive stretching produces arbitrarily fine scale structure!

now, point is that for $\nu \neq 0$
$Re, Pe \gg 1$,
instead of arbitrarily fine scale structure,
must have: $\omega(\psi) \to \text{const}$ as $t \to \infty$ — i.e. small ν ⇒ global behavior

⇒ i.e. finite ν at large Re ⇒
vorticity homogenization!, $\omega \to \text{const}$ within Co

⇒ highly singular behavior!
$\nu = 0$ → Euler Eqn. (2D) → $\omega = \omega(\psi)$ solves

$\nu \neq 0$ → large Re 2D Navier-Stokes Eqn. → $\omega = \text{const}$ solves

note contrast!

Issues:

→ how long to homogenization? (what means asymptotic)

→ where is $\nabla\omega \neq 0$ ⇒ boundary layer thickness?

→ analogy in MHD? — Flux Expulsion

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J} \qquad \begin{aligned} \underline{V} &= \nabla\phi \times \hat{z} \\ \underline{B} &= \nabla A \times \hat{z} \end{aligned}$$

$$-\frac{1}{c}\partial_t \underline{A} - \nabla\phi + \frac{(\nabla\phi \times \hat{z})}{c} \times (\nabla A \times \hat{z}) = \eta \underline{J}$$

$\hat{z} \cdot (\quad)$

$$\Rightarrow -\frac{1}{c}\partial_t A_z - \cancel{\nabla\phi \cdot \hat{z}} + \hat{z} \cdot \left[\cancel{(\nabla\phi \times \hat{z}) \cdot \hat{z}}\right] \nabla A$$

$$- \frac{(\nabla\phi \times \hat{z}) \cdot \nabla A}{c} = \eta \underline{J}$$

$$\therefore \quad \partial_t \underline{A} + \nabla\phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A$$

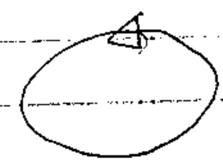
$$\Rightarrow \text{2D convection} \begin{cases} \nabla \cdot \underline{V} = 0 \\ \eta \neq 0 \end{cases}$$

⇒ expect $\nabla A \underset{t \to \infty}{=} 0$, except boundaries

→ analogy is "flux expulsion"

E.) Prandtl - Batchelor Theorem

* G. Batchelor, JFM 1 177 (1956) (posted)
P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)
JFM 133 133 '83
J. Pedlosky, "Ocean Circulation Theory"
see also Springer 1996, esp. 3.8.



Prandtl - Batchelor Theorem

Thm: Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline C_0. Then, if diffusive dissipation,

i.e. $\partial_t \omega + \nabla\phi \times \hat{z} \cdot \nabla\omega = \nabla \cdot (\nu \nabla \omega)$

then vorticity → uniform (homogenization), as $t \to \infty$, within C_0.

N.B.: ① finite ν ⇒ radically different final state

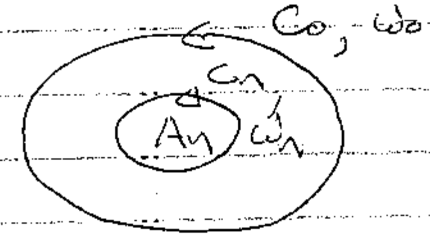
② no comment on "how long"?

- $\underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\omega = \underline{\nabla} \cdot \nu \underline{\nabla}\omega$

for stationarity

[note $t \to \infty$ before $\nu \to 0$]

- choose arbitrary closed C_n within C_0. Here C_n a streamline



n.b. - assume simply connected region, i.e. no holes
- stationarity $\Rightarrow$ ω constant along streamlines

$\therefore$ $\omega \to \omega_0$ on C_0 (ultimately C_0 sets b.c.)
$\omega \to \omega_n$ on C_n

if A_n is area enclosed by C_n;

$$\int_{A_n} d^2x \; \underline{v} \cdot \underline{\nabla}\omega = \int_{A_n} d^2x \; \underline{\nabla} \cdot \left(\nu \underline{\nabla}\omega\right)$$

but

$$\int_{A_n} d^2x \; \underline{v} \cdot \underline{\nabla}\omega = \int d^2x \; \underline{\nabla} \cdot [\underline{v}\omega]$$

$$= \int_{C_n} dl \; \hat{n}_{c_n} \cdot (\underline{v}\omega)$$

Now

$\mathbf{V}$ → streamline, C_n

$\hat{n}_{C_n}$ → normal to streamline C_n

so $$\int_{C_n} d\ell \left(\hat{n}_{C_n} \cdot \mathbf{V}\right) \omega = 0$$

as $\underline{V}$ is along streamline

$$0 = \int_{A_n} d^2x \; \underline{\nabla} \cdot \left(\nu \, \underline{\nabla} \omega\right)$$

$$= \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \omega$$

now, in stationary state, must have
$\omega \to$ const along streamline

$\therefore \quad \omega = \omega(\phi)$

so $\omega_{C_n} = \omega(\phi_n)$

$$0 = \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \phi_n \, \frac{d\omega}{d\phi_n}$$

$$= \nu \frac{d\omega}{d\phi_n} \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \phi_n$$

but $\Gamma = \int d\underline{l}\cdot\underline{v}$

$$= \int d\underline{l}\cdot(\underline{\nabla}\phi\times\hat{z})$$

$$= \int(\hat{z}\times\vec{n})\cdot(\underline{\nabla}\phi\times\hat{z})$$

$$= -\int dl\,(\underline{\nabla}\phi\cdot\vec{n}) = -\int dl\,(\underline{\nabla}\phi\cdot\vec{n})$$

$$0 = \nu\,\frac{\partial\omega}{\partial\phi_n}\,\Gamma_n$$

$$\partial\omega/\partial\phi_n = 0$$

but ϕ_n arbitrary $\Rightarrow$ $\partial\omega/\partial\phi = 0$, all ϕ

arbitrary $\Rightarrow$ no variation from line to line

$\Rightarrow$ $\boxed{\omega \text{ homogenized}}$!

so, expect $\nabla\omega$ large at bounding contour C_0.

$\nabla\omega \neq 0$, within $\Rightarrow$ $\nabla\omega$ held at boundary

Some Comments:

⇒ Homogenization theory looks 'magical' → caveat emptor!

i.e.

1.) note assumptions of

$t \to \infty$ ⇒ time asymptopia

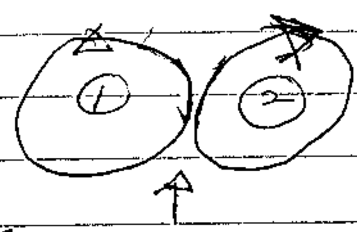
$q = q(\phi)$ ⇒ concentric streamlines

how long to achieve configuration?

2.) simply connected domain → annulus?

3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → (interference of boundary layers?)



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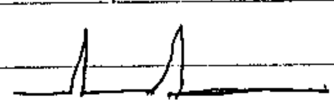


⇒ straining interaction

⇒ ② 'strips' ①, etc.

4.) Key Assumptions:

→ {closed, bounding streamline
{viscous dissipation

i.e. can envision:

→ exact streamline, molecular viscosity

or

→ coarse-grained streamline, eddy viscosity

⇒ correspond to homogenization of

→ total vorticity

→ mean/coarse-grained vorticity

→ time scales different

→ $\dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \Rightarrow Re \gg 1$ $\leftarrow L \rightarrow$

- so establish concentric circulation lines

then

- diffuse across to homogenize → but slow ??

$$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$$

i.e. $Re \gg 1$

or equivalently $\frac{VL}{D} >> 1$

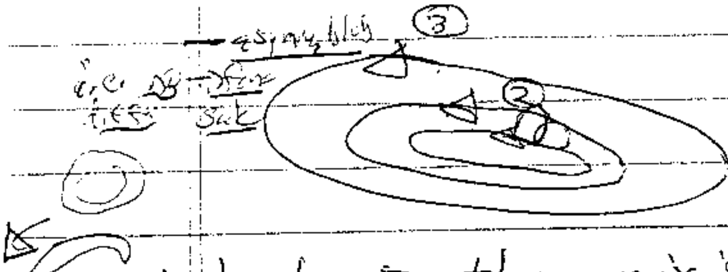
i.e. $(Re)_{\substack{eff \\ cell}} >> 1$ ~~Pe~~ $Pe >> 1$

related: - essential idea is that Ω constant along streamlines established on fast ($\sim \tau_c$) scale
few

- dissipation homogenizes on slower ($\sim \tau_D$) time scale (but this is slow...)

→ What are the time scales? — { how resolve slow time scale problem

- useful to consider differentially rotating, sheared flow within closed pattern



$V_{①} \neq V_{②} \neq V_{③}$

need blob with finite l_y

What is the mixing time scale?

finer, finer scale ...

Key: synergism between { shear, diffusion !

c.f. { H. Biglari, P.H. Diamond, P.W. Terry, Phys Fluids (B2), 1, 1990

~~(first noted by G.I. Taylor)~~

mixing — Shear Dispersion

i.e. compare 

time ← L →

radial diffusion

(a) pure diffusion

$1/\tau \sim D/L^2$

$\boxed{\langle \delta r^2 \rangle \sim D t}$ → usual diffusion

$D_r \sim \langle \tilde{V}_r^2 \rangle \tau_c$

any radial scattering process (unspecified)

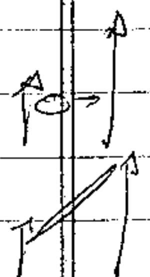
with

(b) diffusion + shear hybrid

$V' \sim V/L$

shear

now $\frac{dr}{dt} = \tilde{V}_r$ → random walk



$r\theta = y$

$\frac{dy}{dt} = V_{0y}(r)$ → streaming

shear

$\therefore \frac{d}{dt}\delta y = \left(\frac{\partial V_{0y}}{\partial r}\right)\delta r$

$\delta y = \int \left(\frac{\partial V_{0y}}{\partial r}\right)\delta r \, dt$

$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \underbrace{\langle \delta r^2 \rangle t^2}$

$\langle \delta r^2 \rangle \sim D_r t$

$\Rightarrow \boxed{\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 D_r t^3}$ → ~~shear dispersion~~ hybrid decorrelation $\langle \delta y^2 \rangle \sim t^3$

scale of comparison

$$\langle \delta y^2 \rangle \sim L_y^2 \Rightarrow \underline{\text{arbitrary}}$$

$$1/\tau_{mix} = \left(\left(\frac{\partial V_y}{\partial x}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \left(\left(\frac{V_0}{L_D}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \frac{V_0}{L_y^2} \left(\frac{D}{V L_D} \right)^{1/3}$$

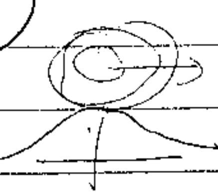
$$\therefore \quad 1/\tau_{mix} \sim \frac{1}{\tau_c} \underline{(Re)}^{-1/3}$$

$Re \gg 1$ by construction $\rightarrow$ consistent ✓

so have:

$\rightarrow$ <u>mixing / homogenization</u> on <u>hybrid</u> <u>time scale</u> $\rightarrow$ <u>time to come to ⊙ symmetric state</u>

$$1/\tau_{mix} = 1/\tau_c \left(\tau_c / \tau_D \right)^{1/3}$$



only radial diffsn possible

$$\Rightarrow \frac{1}{\tau_c} > \frac{1}{\tau_{mix}} > \frac{1}{\tau_D}$$

<u>time to uniformize</u> vs τ_D

$\Rightarrow$ $\boxed{\text{PV homogenization most relevant to closed eddys with sheared rotation}}$

Some Points

i.) Time scales

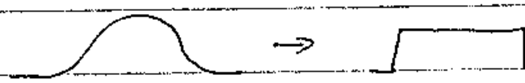
have Re, Pe $\gg 1$ $\Rightarrow \frac{\tau_D}{\tau_c} \gg 1$

but $\tau_{mix} \sim Re^{1/3} \tau_c \sim \frac{\tau_D}{Re^{2/3}}$

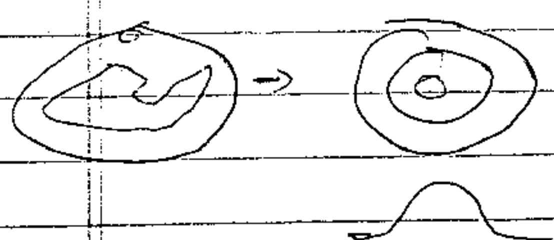
$\tau_{mix} < \tau_D \rightarrow$ time to homogenize

$\downarrow$

time to establish $\sim$ azimuthally symmetric state



i.e.



but radially profiled

ii.) Point of theorem is global impact of small dissipation.

iii.) Interesting to note that P-B theorem applies to both active, passive scalar.

ii) Observe: all that is really required for applicability of theory is:

- → incompressible advection - 2D: $\nabla\phi\times\hat{z}\cdot\nabla$
- → closed streamline → {fine, coarse}
- → diffusive dissipation → {molecular, eddy}

i.e. can apply to magnetic potential, as noted previously, i.e.

2D MHD $\underline{v} = \nabla\phi\times\hat{z}$, $\underline{B} = \nabla A\times\hat{z}$

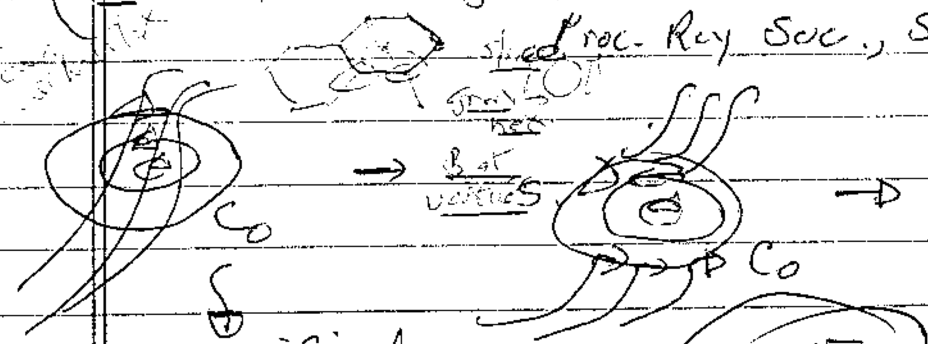
$$\frac{\partial A}{\partial t} + \underline{v}\cdot\nabla A = \eta\nabla^2 A$$

$E + v\times B = \eta J$

$-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi + (\nabla\phi\times\hat{z})\times(\nabla A\times\hat{z}) = \eta J$

→ Famous problem of Flux Expulsion

(i.e. N. Weiss, Proc. Roy Soc., Series A 293, 310, 1966)



→ $\nabla A \to 0$ within

→ B expelled to boundary (i.e. $B = \nabla A\times\hat{z}$)

→ $B \sim 0$ in cell.

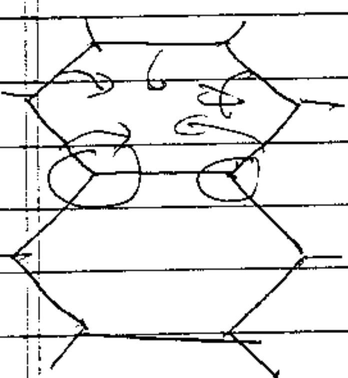
specified convective cell

(has magnetoconvection as aim)

→ obvious that above argument can be recycled, so

$$\delta A_n/\delta\phi_n = 0, \quad \text{for all } C_n \text{ in } C_0.$$

i.e. top view → solar granulation

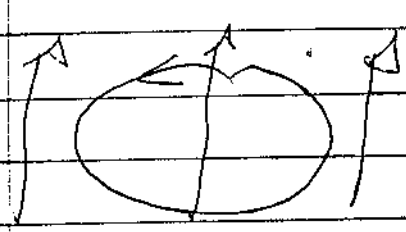


~ hexagonal pattern

~ field strongest at vertices

~ expulsion

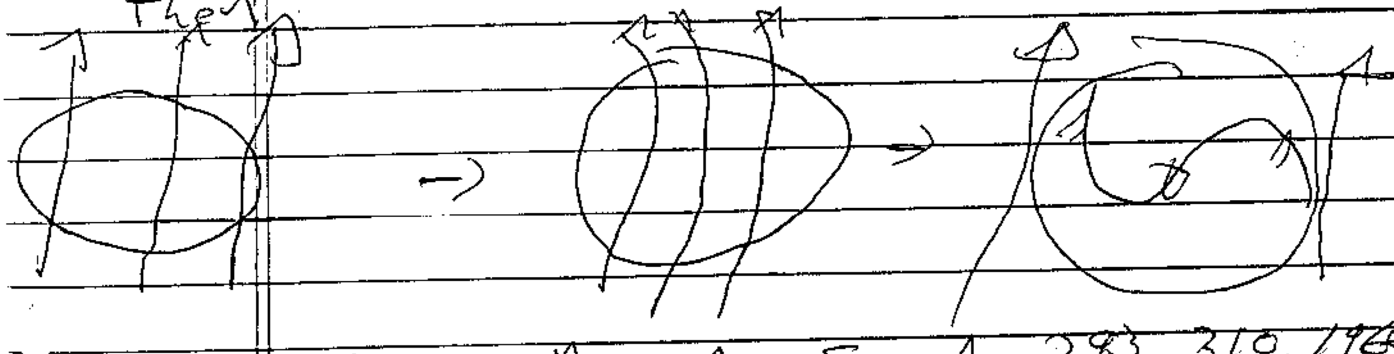
suggests → side view



→ toy problem of flux expulsion

→ i.e. cell in uniform field

then



see Weiss, Proc. Roy Soc A 293, 310. 1966

also Moffatt, 3.7 – 3.10

→ here, requirement is $R_m = \frac{LV}{\eta} \gg 1$

and

$$1/\tau_{homog} \sim 1/\tau_{mix} \sim \frac{V_0}{L_0}(R_m)^{-1/3}$$

time scale for flux expulsion.

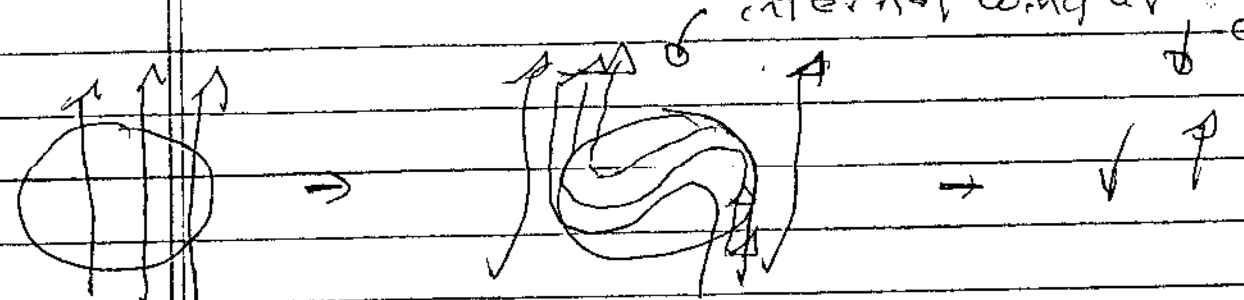
→ physically, can see relevant time scale by noting:

→ wind up must conserve volume/mass

→ wind up must conserve flux

→ irreversibility sets in when $R_m^{local} \sim 1$

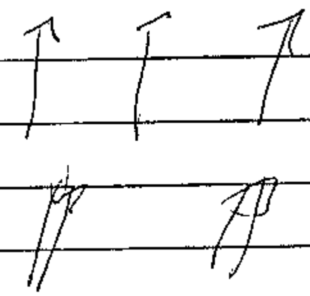
⇒ dissipation of field matches drive by wind-up



internal cancellation → { field expulsion, flux homogenization } so

- field stretched / compressed in n-wards

$n\,l = L_0$



$l = L_0/n$

- flux (vertical) conserved, so

so $B_0 L \sim \ell B$ (B ↑ upon flux conservation/winding)

⇒ $B \sim n B_0$

⇒ now, expect freezing-in lost when compressed field

$$(R_m)_{eff} \sim 1 \Rightarrow \frac{V B_0}{L_0} \sim \eta \frac{B}{\ell^2} \sim \eta \frac{n B_0}{L_0^2/n^2}$$

⇒

$$n^3 \sim \left(\frac{V_0 L_0}{\eta}\right) \Rightarrow \boxed{n \sim R_m^{1/3}}$$

# of turns to render boundary diffusive

{ ~~thickness of B layer~~

$\frac{V}{L} B \sim \frac{\eta B}{\delta^2}$ $\quad \frac{\delta}{L} \sim 1/\sqrt{R_m}$

but: – diffusion in boundary layer ⇔ homogenization within

so

– $n \sim R_m^{1/3}$ ⇒ # turns for homogenization

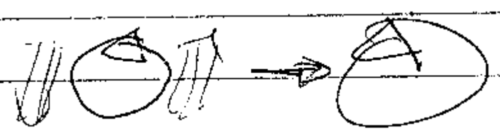
– $\tau_{hom} \sim \tau_0 R_m^{1/3}$ → time ✓

agree above

⇒ expect flux expelled from closed cell
⇒ field strongest at boundary

→ possibly explain why strongest cells in 2D

magneto convection often
reach a state independent of $\underline{B}$
while neighbors quenched.



Now: ⇒ why care about this?
Why homogenization important

→ [illegible], Newman

① identifies a trend; i.e. in spirit of
Taylor Theory (E_{mag} minimized s/t

$\int \underline{A} \cdot \underline{B} \, d^3x$ conserved), homogenization

theory identifies a trend, i.e.

if f – conserved locally by
2D flow, $\underline{\nabla} \cdot \underline{v} = 0$
– ~~diffused~~
– enclosed

⇒ f homogenized

② trend applies to
- non-passive → vorticity
- passive → A, C

In particular,

③ trend severely constrains form of vorticity flux, flow evolution

i.e. zonal flows ⇒ 2D closed streamline flows ...

→ Do zonal flows tend homogenize PV?

→ if noise (i.e. emission), what scale selected?

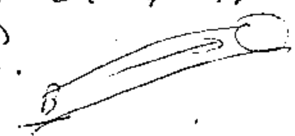
How translate reconnection into global trend?
Consequence?

→ Magnetic Helicity

- another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A}\cdot\underline{B}$$

V is taken to be the volume of a 'flux tube'.



- what, yet another invariant?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x \; A\cdot\underline{\nabla}\times\underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a $\boxed{\text{pseudo-scalar}}$

∴ has orientation or "handedness"...

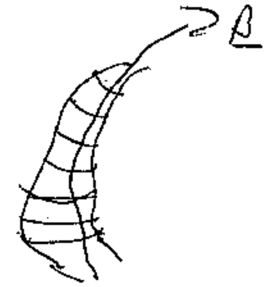
Proceed via:

- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x\, \underline{\nabla}\chi \cdot \underline{B}$$



$$= K + \int_V d^3x\, \underline{\nabla}\cdot(\underline{B}\,\chi)$$

$= 0$, to surface term. $\left\{ \underline{B}\cdot\hat{n} = 0 \right.$ on surface of tube

Now, consider a blob of MHD fluid in motion

can show $\underbrace{\dfrac{dK}{dt} = 0}_{\text{ideal}}$.

$$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = \eta\,\underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{V}\times\underline{\nabla}\times\underline{A} - c\nabla\phi - c\eta\,\underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V}\cdot\underline{\nabla}\,\underline{B} + \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V} + \eta\,\nabla^2\underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt}\int_V d^3x\,(\underline{A}\cdot\underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt}\cdot\underline{B} + \underline{A}\cdot\frac{d\underline{B}}{dt}\right) + \int \underline{A}\cdot\underline{B}\,\frac{d}{dt}\,d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + \left(\underline{V}\cdot\underline{\nabla}\,\underline{A}\right)\cdot\underline{B} + \underline{A}\cdot\frac{\partial \underline{B}}{\partial t} \right.$$

$$\left. + \underline{A}\cdot\left(\underline{V}\cdot\underline{\nabla}\,\underline{B}\right)\right) + \underline{A}\cdot\underline{B}\;\underline{\nabla}\cdot\underline{V} \Big)$$

where $\frac{d}{dt}\, d^3x = \underline{\nabla}\cdot\underline{V}$

i.e. $\frac{d}{dt}\, dV = \frac{d}{dt}\, d\underline{\sigma}\cdot d\underline{\ell} + d\sigma \cdot \frac{d}{dt} d\underline{\ell}$

$$= - d\underline{\ell}\cdot\underline{\nabla}\,\underline{V}\cdot d\underline{\sigma} + (\underline{\nabla}\cdot\underline{V})(d\underline{\sigma}\cdot d\underline{\ell}) + d\underline{\ell}\cdot\underline{\nabla}\,\underline{V}\cdot d\underline{\sigma}$$

$$= \underline{\nabla}\cdot\underline{V}\; d^3\cancel{x}$$

s.t. and $\underline{B}\cdot\hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \left[\left(\underline{B}\cdot\cancel{\underline{V}}\times\underline{B} - c\,\cancel{\underline{B}}\cdot\underline{\nabla}\phi - c\eta\,\underline{J}\cdot\underline{B} \right) \right.$$

$$\left. + \underline{A}\cdot\left(\underline{\nabla}\times(\underline{V}\times\underline{B})\right) + \underline{\nabla}\cdot\left((\underline{A}\cdot\underline{B})\,\underline{V}\right) + \underline{A}\cdot\eta\nabla^2\underline{B} \right]$$

where $\underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B}) + \underline{B}\cdot(\underline{V}\cdot\underline{\nabla}\underline{A}) + \underline{A}\cdot\underline{B}\;\nabla\cdot\underline{V} = \nabla\cdot\left(\underline{V}\,\underline{A}\cdot\underline{B}\right)$

$$\frac{dK}{dt} = \int d^3x \left[\nabla\cdot\left((\underline{A}\cdot\underline{B})\,\underline{V}\right) + \underline{\nabla}\cdot\left((\underline{V}\times\underline{B})\times\underline{A}\right) + (\underline{V}\times\cancel{\underline{B}})\cdot(\underline{\nabla}\times\underline{A}) \right.$$

$$\left. - c\eta\,\underline{\sigma}\cdot\underline{B} - \eta(\underline{A}\cdot\underline{\nabla}\times\underline{J}:)\,c \right]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta(\underline{A}\times\underline{J})\right] - c\eta\,\underline{J}\cdot\underline{B} - c\eta\,\underline{J}\cdot\underline{B}\right\}$$

$$= \int d\underline{S}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\,\underline{A}\times\underline{J}\right]$$

$$- 2\int d^3x\left[c\eta\,\underline{J}\cdot\underline{B}\right]$$

$$= \int d\underline{S}\cdot\left[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \cancelto{0}{(\underline{A}\cdot\underline{v})\underline{B}}\right] - c\eta\int d\underline{S}\cdot\underline{J}\times\underline{A}$$

$\underline{B}\cdot\underline{n} = 0$, on tube

$$- 2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$$= -\int c\eta\, d\underline{S}\cdot\left[\underline{\nabla}\underline{B}\cdot\underline{A} - \cancelto{0}{\underline{A}\cdot\underline{\nabla}\underline{B}}\right] - 2c\eta\int d^3x\,\underline{J}\cdot\underline{B}$$

$$= -2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta\int d^3x(\underline{J}\cdot\underline{B})}$$

and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)
↳ why? $\sigma - \nu$ $\underline{J} \sim 1/\eta^{1/2}$

$\therefore$ Magnetic Helicity is conserved in ideal MHD

$\frac{dK}{dt} \sim \eta^{1/2}$

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

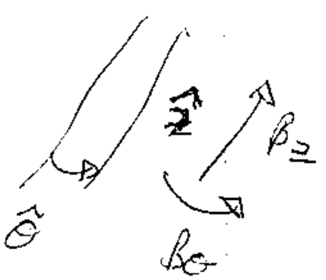
$\underline{B} \cdot \underline{\nabla} \alpha = 0$, $\underline{B} \cdot \underline{\nabla} \psi = 0$

$B_\theta \frac{m}{r} - \frac{n B_z}{R} = \frac{B_z}{R}\left(\frac{m}{q(r)} - n\right)$

safety factor
⇓

interesting to note: $q(r) = \frac{r B_z}{R B_\theta(r)} = \frac{1}{R\mu(r)}$

$\frac{m}{n} \to$ pitch pert.
$q(r) \to$ pitch line



$\mu(r) = \frac{B_\theta(r)}{r B_z}$ → field line pitch.

(lengthscale over which winding varies)

cylindrical plasma → $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r}\int_0^r r B_z \, dr$

$$A_z = -\int_0^r B_\theta \, dr$$

so $$\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\,dr - B_z\int_0^r B_\theta\,dr$$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\,dr - B_z\int_0^r B_\theta\,dr$$

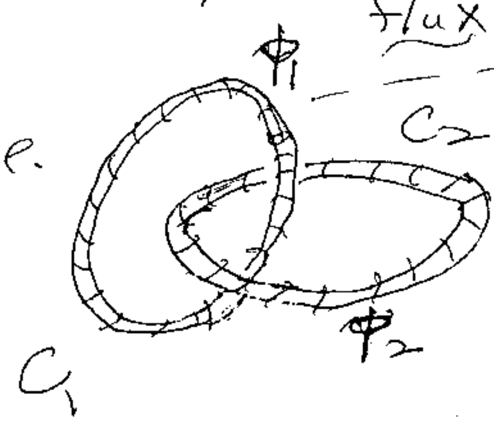
$$\underline{A}\cdot\underline{B} = B_z\left[\mu\int_0^r \frac{B_\theta}{\mu}\,dr - \int_0^r B_\theta\,dr\right]$$

$= 0$ for constant μ !

∴ non-zero helicity requires $\mu = \mu(r)$
i.e. – pitch varies with radius (minor)
⇒ $\boxed{\text{magnetic shear}}$!

– physically → helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux $\underline{dA}$

$$\Phi = \int \underline{dA}\cdot\underline{B} = \phi_1$$

X-section area; const

tube 2: $\Phi = \phi_2$

field in loops, only

Now, for volume V_1 of tube I

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\; d^3x = \oint_{\substack{C_1 \\ \text{along loop}}} d\ell \int_{\substack{A_1 \\ \text{x-sect area of loop}}} \underline{dS}\; \underline{A}\cdot\underline{B}$$

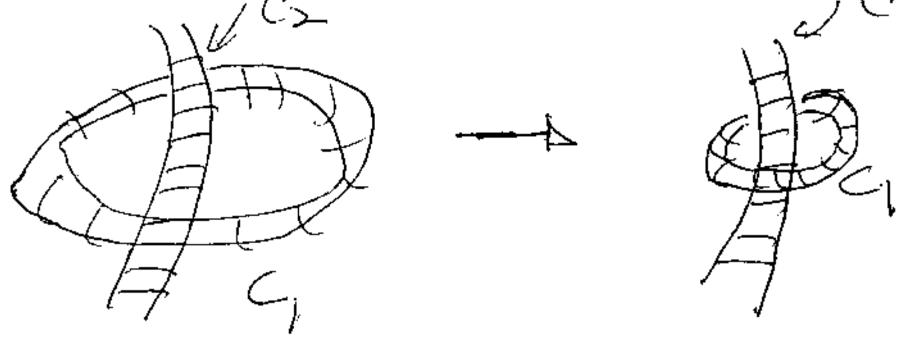
$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\; dA$$

$\underline{B}$ in tube $\Rightarrow$ flux thru x-section

$\therefore$ A encircles 2

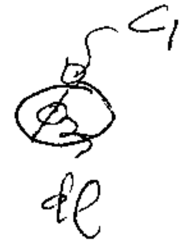
$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops, till encircles C_2



re-oriented

$\rightarrow$ in x section;



but $\displaystyle\int_{C_1} A\cdot d\ell = \int_{A\text{ enclosed}} B\cdot dS = \Phi_2$

so ... $K_1 = \phi_1 \phi_2$ $\Rightarrow$ product of fluxes

similarly $K_2 = \phi_2 \phi_1$

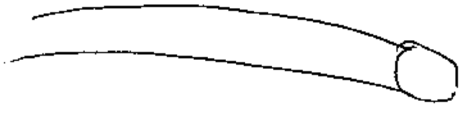
$\therefore \quad K = 2\phi_1 \phi_2$

if $\underline{\underline{n}}$ windings $\quad K = K_1 + K_2 = \pm 2n\phi_1 \cdot \phi_2$.

$\Rightarrow$ helicity is measure of self-linkage of magnetic configuration.

Why care $\rightarrow$ Taylor Conjecture (1974) (J. B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP  ~ toroid ~ toroidal current

well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$ $\Leftrightarrow$ force free

$\Rightarrow$ why so robust ?, especially since RFP so turbulent

– Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point – helicity conserved in flux tubes, to η
– toroidal plasma $\leftrightarrow$ many small tubes

etc.

– recall Sweet–Parker model: magnetic reconnection / resistive dissipation effective on small scales.
i.e. $\tau_R \sim \tau_A/\sqrt{R_m} \sim L^{1/2}$ $\quad R_m = v_A L/\eta$ — turbulence accesses small scale

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\int_V \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow \text{conserved.}$$

V – plasma volume

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\, \frac{B^2}{8\pi} + \lambda \int_V d^3x\, \underline{A}\cdot\underline{B}\right] = 0 \qquad \boxed{\nabla\times\underline{B} = \lambda\underline{B}}$$

$$\delta A \cdot (\nabla\times A)$$

$$\delta(\nabla\times A)^2 =$$

$$(\nabla\times A)\cdot\delta\nabla\times A = (\nabla\times A)\cdot\nabla\times\delta A = -(\nabla\times\nabla\times A)\cdot\delta A$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity,

Comments:

→ it works! – indeed amazingly well – for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay
- → energy cascades → small scale
- helicity cascades → large scale (less dissipation)

- relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing.