

Physics 222 UCSD/225b UCSB

Lecture 10

Chapter 14 in H&M.

Chapter 14 Outline

- Reminder of Lagrangian formalism
 - Lagrange density in field theory
- Aside on how Feynman rules are derived from Lagrange density.
- Reminder of Noether's theorem
- Local Phase Symmetry of Lagrange Density leads to the interaction terms, and thus a massless boson propagator.
 - Philosophically pleasing ...
 - ... and require to keep theory renormalizable.
- Higgs mechanism to give mass to boson propagator.

Reminder of Lagrange Formalism

- In classical mechanics the particle equations of motion can be obtained from the Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

- The Lagrangian in classical mechanics is given by:

$$L = T - V = E_{\text{kinetic}} - E_{\text{potential}}$$

Lagrangian in Field Theory

- We go from the generalized discrete coordinates $q_i(t)$ to continuous fields $\phi(x,t)$, and thus a Lagrange density, and covariant derivatives:

$$L(q, \dot{q}, t) \rightarrow L(\phi, \partial_\mu \phi, x_\mu)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \rightarrow \frac{\partial}{\partial x_\mu} \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) - \frac{\partial L}{\partial \phi} = 0$$

Let's look at examples (1)

- Klein-Gordon Equation:

$$\frac{\partial}{\partial x_{\mu}} \left(\frac{\partial L}{\partial (\partial_{\mu} \phi)} \right) - \frac{\partial L}{\partial \phi} = 0$$

$$L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2$$

$$\partial_{\mu} \partial^{\mu} \phi + m^2 \phi = 0$$

Note: This works just as well for the Dirac equation. See H&M.

Let's look at examples (2)

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial L}{\partial A_\nu} = 0$$

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu$$

Maxwell Equation:

$$\partial_\mu F^{\mu\nu} = j^\mu$$

$$-\frac{\partial L}{\partial A_\nu} = j^\nu$$

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(-\frac{1}{4} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \right) =$$

$$= -\frac{1}{2} g^{\alpha\alpha} g^{\beta\beta} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left((\partial_\alpha A_\beta)^2 - (\partial_\alpha A_\beta \partial_\beta A_\alpha) \right) = -\frac{1}{2} 2 (\partial^\mu A^\nu - \partial^\nu A^\mu) =$$

$$= -F^{\mu\nu}$$

Aside on current conservation

- From this result we can conclude that the EM current is conserved:

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\mu$$

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) =$$

$$= \partial_\mu \partial^\mu \partial_\nu A^\nu - \partial_\nu \partial^\nu \partial_\mu A^\mu = \partial_\mu \partial^\mu (\partial_\nu A^\nu - \partial_\mu A^\mu) = 0$$

- Where I used:
 $\partial_\mu \partial^\mu = \partial_\nu \partial^\nu$
 $\partial_\nu A^\nu = \partial_\mu A^\mu$

Aside on mass term

- If we added a mass term to allow for a massive photon field, we'd get:

$$(\partial_{\mu}\partial^{\mu} + m^2)A^{\nu} = j^{\nu}$$

This is easily shown from what we have done.
Leave it to you as an exercise.

Feynman rules from the Lagrangian

- We associate vertex factors and propagators with the various terms in the lagrangian.
 - Propagators come from terms that are quadratic in the fields.
 - All other terms in lagrangian are interaction terms, and are thus associated with vertices.
- You learn how to do this rigorously in QFT. Here, we'll be happy enough to simply state the Feynman rules as needed. So let's not waste time on this.

Noether's theorem

- Every symmetry of the lagrangian has a corresponding quantity that is conserved by the interactions described by the Lagrangian.
- Example: Electron and global phase shift.

The symmetry group of all such transformations is called U(1). U(1) is an abelian group.

$$\left\{ \begin{array}{l} \psi(x) \rightarrow e^{i\alpha} \psi(x) \\ \partial_\mu \psi(x) \rightarrow e^{i\alpha} \partial_\mu \psi(x) \\ \overline{\psi(x)} \rightarrow e^{-i\alpha} \overline{\psi(x)} \end{array} \right.$$

L is invariant under this transformation!

$$L = i\overline{\psi}\gamma_\mu \partial^\mu \psi - m\overline{\psi}\psi$$

Conserved current from L

- We will show in a second that symmetry of the Lagrangian with respect to a given group implies a conserved current for each of the generators of the group.
- We will show explicitly how to derive the form of the current from the Lagrangian by working through an infinitesimal transformation.
- We'll do so for $U(1)$ because that's easiest algebraically. The idea works the same for any other symmetry group.

Infinitesimal U(1) Transformation

$$\psi(x) \rightarrow (1 + i\alpha)\psi(x)$$

$$L = i\bar{\psi}\gamma_{\mu}\partial^{\mu}\psi - m\bar{\psi}\psi$$

$$0 = \delta L$$

$$\delta L = \frac{\partial L}{\partial \psi} \delta\psi + \frac{\partial L}{\partial(\partial_{\mu}\psi)} \delta(\partial_{\mu}\psi) + \frac{\partial L}{\partial \bar{\psi}} \delta\bar{\psi} + \frac{\partial L}{\partial(\partial_{\mu}\bar{\psi})} \delta(\partial_{\mu}\bar{\psi})$$

$$0 = i\alpha \left[\frac{\partial L}{\partial \psi} \psi + \frac{\partial L}{\partial(\partial_{\mu}\psi)} \partial_{\mu}\psi \right] - i\alpha \left[\frac{\partial L}{\partial \bar{\psi}} \bar{\psi} + \frac{\partial L}{\partial(\partial_{\mu}\bar{\psi})} \partial_{\mu}\bar{\psi} \right]$$

Next, we reorder terms to use Lagrange Equation.

Before reordering, consider the following:

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \psi \right) = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) \psi + \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) \partial_\mu \psi$$

$$\left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) \partial_\mu \psi = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \psi \right) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) \psi$$

And the same for the “anti-spinors”.

This is part of the current.

This is part of lagrange equation.

=0 because of Lagrange Equation

$$0 = i\alpha \left[\frac{\partial L}{\partial \psi} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) \right] \psi + i\alpha \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \psi \right) + \dots$$

Only current piece left

$$0 = i\alpha \left[\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \psi \right) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \bar{\psi})} \bar{\psi} \right) \right] \Rightarrow \partial_\mu j^\mu = 0$$

Conserved current as result of global phase invariance!

$$j^\mu = \frac{ie}{2} \left[\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \psi \right) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \bar{\psi})} \bar{\psi} \right) \right] = -e \bar{\psi} \gamma^\mu \psi$$

Conserved charge and current

$$\partial_\mu j^\mu = 0 \quad \longleftarrow \text{Conserved current}$$

$$j^\mu = \frac{ie}{2} \left[\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \psi \right) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \bar{\psi})} \bar{\psi} \right) \right] = -e \bar{\psi} \gamma^\mu \psi$$

$$Q = \int d^3x \quad j^0 \quad \longleftarrow \text{Conserved charge}$$

Other Lagrangians of interest

- We have now shown that the Lagrangian that gives us Dirac Equation has a U(1) symmetry that gives us charge conservation, as well as a conserved current.
- We can do the same for a Lagrangian for a complex scalar field:

$$L = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi$$

Conserved current:
$$j^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

Local Phase Invariance

- Next, we look at what happens when we impose local phase invariance, rather than global phase invariance.

$$\text{Local Gauge transformation} \quad \left\{ \begin{array}{l} \psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \\ \partial_\mu\psi(x) \rightarrow e^{i\alpha(x)}\partial_\mu\psi(x) + ie^{i\alpha(x)}\psi\partial_\mu\alpha(x) \\ \overline{\psi}(x) \rightarrow e^{-i\alpha(x)}\overline{\psi}(x) \end{array} \right.$$

- We will find that this leads us to the interaction terms for the bosonic field that provides the propagator of the interaction.

Let's work this through:

$$L(\psi) = i\bar{\psi}\gamma_{\mu}\partial^{\mu}\psi - m\bar{\psi}\psi$$

$$L(e^{i\alpha(x)}\psi) = L(\psi) - \bar{\psi}\gamma_{\mu}\psi\partial^{\mu}\alpha$$

If we add to the Lagrangian a piece that transforms appropriately, then we may regain the desired symmetry.

$$A^{\mu} \rightarrow A^{\mu} + \frac{1}{e}\partial^{\mu}\alpha$$

$$L(\psi) = \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - m)\psi + e\bar{\psi}\gamma_{\mu}\psi A^{\mu}$$

$$L(e^{i\alpha(x)}\psi) = L(\psi) - \bar{\psi}\gamma_{\mu}\psi\partial^{\mu}\alpha + \bar{\psi}\gamma_{\mu}\psi\partial^{\mu}\alpha = L(\psi)$$

Reminder of E&M

- You probably recall from E&M, that the vector potential A is not fully determined by the E&B fields. Instead, we have exactly the desired invariance:

$$A^\mu \rightarrow A^\mu + \frac{1}{e} \partial^\mu \alpha$$

- Where $\alpha(x)$ is an arbitrary scalar field.
- What seemed at best obscure in E&M is now a necessary condition for local phase symmetry, or local U(1) symmetry.

Complete QED Lagrangian

- Once we consider A as the physical photon field, it becomes obvious that we ought to add its kinetic energy to complete the Lagrangian:

$$L(\psi) = \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - m)\psi + e\bar{\psi}\gamma_{\mu}\psi A^{\mu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Massive Gauge Fields

- In principle, we might want to add a mass term as well. However, it's easy to see that this destroys local phase invariance:

$$A^\mu \rightarrow A^\mu + \frac{1}{e} \partial^\mu \alpha$$

$$m^2 A_\mu A^\mu \rightarrow m^2 \left(A_\mu + \frac{1}{e} \partial_\mu \alpha \right) \left(A^\mu + \frac{1}{e} \partial^\mu \alpha \right) \neq m^2 A_\mu A^\mu$$

There are clearly lot's of terms that don't cancel!

Summary on local U(1) Symmetry

- By imposing local phase invariance on the free fermion Lagrangian we are led to the interacting field theory of QED.
- Gauge symmetry of the 4-vector potential, which seemed at best a curiosity in classical E&M, has become one of the most basic and essential ingredients to allow local U(1) in QED.

Non-Abelian Case -> QCD

- I will skip chapter 14.4 in the interest of time. It's just tedious, and doesn't add much fundamentally new.
- I encourage you to at least read through the chapter 14.4 carefully!

Massive Gauge Bosons

- We saw that adding a mass term for the gauge bosons violates the $U(1)$ symmetry.
- However, it moreover spoils renormalizability of the theory.
- While this is beyond the scope of this course, it's worth contrasting the propagators for a massive vs massless field, and show how loop integrals diverge, and perturbation theory thus makes no sense.

Massless vs Massive

$$\frac{g_{\mu\nu}}{q^2} \qquad \frac{-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}}{q^2 - M^2}$$

Loop integrals are of the general form:

$$\int d^4q \text{ (propagators)}$$

The $1/q^2$ behavior of the massless propagator keeps such integrals finite. The massive propagator has a “ $1/M^2$ ” behaviour, thus leading to divergent loop integrals.

