chapter 24 Gauss's law

Problem

32. Repeat Problem 26, assuming that Fig. 24-45 represents the cross section of a long, thick-walled pipe. Now the case $a = 0$ should be consistent with the result of Problem 31 for the interior of the rod.

Solution

Suppose that the pipe is long enough that the line symmetric result in Equation 24-8 can be used in Gauss's law. Then $E = q_{\text{enclosed}}/2\pi\varepsilon_0 r\ell$. For $a < r < b$, $q_{\text{enclosed}} = \rho V = \rho\pi(r^2 - a^2)\ell$, so $E(r) = (\rho/2\varepsilon_0)(r - a^2/r)$. For $a \to 0$, the field inside a uniformly charged solid rod in Problem 31(b) is recaptured.

Problem

66. The volume charge density inside a solid sphere of radius a is given by $\rho = \rho_0 r/a$, where ρ_0 is a constant. Find (a) the total charge and (b) the electric field strength within the sphere, as a function of distance r from the center.

Solution

(a) The charge inside a sphere of radius $r \le a$ is $q(r) = \int_0^r \rho\, dV$. For volume elements, take concentric shells of radius r and thickness dr, so $dV = 4\pi r^2 dr$. Then

$$q(r) = 4\pi\int_0^r \rho r^2\, dr = 4\pi(\rho_0/a)\int_0^r r^3\, dr = \pi\rho_0 r^4/a.$$

For $r = a$, the total charge is $\pi\rho_0 a^3$. (b) For spherical symmetry, Gauss's law and Equation 24-5 give $4\pi r^2 E(r) = q(r)/\varepsilon_0 = \pi\rho_0 r^4/\varepsilon_0 a$, or $E(r) = \rho_0 r^2/4\varepsilon_0 a$.

Problem

72. An infinitely long nonconducting rod of radius R carries a volume charge density given by $\rho = \rho_0(r/R)$, where ρ_0 is a constant. Find the electric field strength (a) inside and (b) outside the rod, as functions of the distance r from the rod axis.

Solution

Line symmetry, Equation 24-8, and Gauss's law give a field strength of $E = \lambda_{\text{enclosed}}/2\pi\varepsilon_0 r$, where $\lambda_{\text{enclosed}} = \int_0^r \rho\, dV$ is the charge within a unit length of coaxial cylindrical surface of radius r, and $dV = 2\pi r\, dr$ is the volume element for a unit length of thin shell with this surface. (a) For $r < R$ (inside rod), $\lambda_{\text{enclosed}} = \int_0^r (2\pi\rho_0/R) r^2\, dr = 2\pi\rho_0 r^3/3R$, hence $E = \rho_0 r^2/3\varepsilon_0 R$. (b) For $r > R$ (outside rod), $\lambda_{\text{enclosed}} = \int_0^R (2\pi\rho_0/R) r^2\, dr = 2\pi\rho_0 R^2/3$, hence $E = \rho_0 R^2/3\varepsilon_0 r$.

CHAPTER 25 ELECTRIC POTENTIAL

Problem

6. A charge of 3.1 C moves from the positive to the negative terminal of a 9.0-V battery. How much energy does the battery impart to the charge?

Solution

$\Delta U_{AB} = q\,\Delta V_{AB} = (3.1\text{ C})(9.0\text{ V}) = 27.9\text{ J}.$

Problem

8. Figure 25-37 shows a uniform electric field of magnitude E. Find expressions for (a) the potential difference ΔV_{AB} and (b) ΔV_{BC}. (c) Use your result to determine ΔV_{AC}.

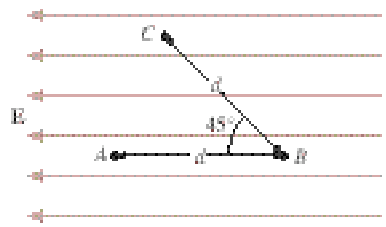


FIGURE 25-37 Problem 8.

Solution

(a) On the line A to B, $\mathbf{d}\ell$ is antiparallel to $\mathbf{E}$, so Equation 25-2 gives $V_B - V_A = -\int_A^B \mathbf{E}\cdot\mathbf{d}\ell = E\int_A^B d\ell = Ed$. (b) The line B to C makes an angle of 45° with $\mathbf{E}$, so $V_C - V_B = -E\int_B^C \cos 45°\, d\ell = -Ed/\sqrt{2}$. (c) Addition yields $V_C - V_A = V_C - V_B + V_B - V_A = Ed(1 - 1/\sqrt{2}) = 0.293Ed$.

Problem

12. Electrons in a TV tube are accelerated from rest through a 25-kV potential difference. With what speed do they hit the TV screen?

Solution

The work done on an electron equals the change in its kinetic energy, $W = e\,\Delta V = \frac{1}{2}mv^2$ (if it starts from rest). Thus,

$$v = \sqrt{2e\,\Delta V/m} = \sqrt{\frac{2(1.6\times10^{-19}\text{ C})(25\times10^3\text{ V})}{(9.11\times10^{-31}\text{ kg})}} = 9.37\times10^7\text{ m/s}.$$

Problem

15. Two large, flat metal plates are a distance d apart, where d is small compared with the plate size. If the plates carry surface charge densities $\pm\sigma$, show that the potential difference between them is $V = \sigma d/\varepsilon_0$.

Solution

The electric field between the plates is uniform, with $E = \sigma/\varepsilon_0$, directed from the positive to the negative plate (see last paragraph of Section 24-6 and Fig. 24-35). Then Equation 25-2b gives $V = V_+ - V_- = -(\sigma/\varepsilon_0)(-d) = \sigma d/\varepsilon_0$ (the displacement from the negative to the positive plate is opposite to the field direction).

Problem

16. An electron passes point A moving at 6.5 Mm/s. At point B the electron has come to a complete stop. Find the potential difference ΔV_{AB}.

Solution

The work-energy theorem (for an electron under the influence of just an electric force) gives $W_{AB} = -q\,\Delta V_{AB} = \Delta K = -K_A$, where W_{AB} is the work done by the electric field (also equal to $-\Delta U_{AB}$), and $K_B = 0$. Thus,

$$\Delta V_{AB} = \frac{K_A}{q} = \frac{m_e v_A^2}{2(-e)} = \frac{(9.11 \times 10^{-31}\text{ kg})(6.5 \times 10^6\text{ m/s})^2}{2(-1.6 \times 10^{-19}\text{ C})} = -120\text{ V}.$$

(To stop an electron, a negative potential difference must be applied.)

Problem

19. The classical picture of the hydrogen atom has a single electron in orbit a distance 0.0529 nm from the proton. Calculate the electric potential associated with the proton's electric field at this distance.

Solution

The potential of the proton, at the position of the electron (both of which may be regarded as point-charge atomic constituents) is (Equation 25-4) $V = ke/a_0$, where a_0 is the Bohr radius. Numerically, $V = (9 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2) \times (1.6 \times 10^{-19}\text{ C})/(5.29 \times 10^{-11}\text{ m}) = 27.2\text{ V}$. (The energy of an electron in a classical, circular orbit, around a stationary proton, is one half its potential energy, or $\frac{1}{2}U = \frac{1}{2}(-e)V = -13.6\text{ eV}$. The excellent agreement with the ionization energy of hydrogen was one of the successes of the Bohr model.)

Problem

21. Points A and B lie 20 cm apart on a line extending radially from a point charge Q, and the potentials at these points are $V_A = 280$ V, $V_B = 130$ V. Find Q and the distance r between A and the charge.

Solution

Since $V_A = kQ/r_A$ and $V_B = kQ/r_B$, division yields $r_B = (V_A/V_B)r_A = (280/130)r_A = 2.15r_A$. But $r_B - r_A = 20$ cm, so $r_A = (20\text{ cm})(2.15 - 1)^{-1} = 17.3$ cm. Then $Q = V_A r_A/k = (280\text{ V})(17.3\text{ cm})/(9 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2) = 5.39$ nC.

Problem

22. What is the maximum potential allowable on a 5.0-cm-diameter metal sphere if the electric field at the sphere's surface is not to exceed the 3 MV/m breakdown field in air?

Solution

For an isolated metal sphere, the potential at the surface is $V = kQ/R$, while the electric field strength at the surface is $kQ/R^2 = V/R$. Thus, $V/R \le 3$ MV/m implies $V \le (3\text{ MV/m})(\frac{1}{2} \times 0.05\text{ m}) = 75$ kV.

$-e(V_\infty - V_{\text{surf}}) = eV_{\text{surf}} = \frac{1}{2}mv^2$. Then $v = \sqrt{2eV_{\text{surf}}/m} = [2(1.6 \times 10^{-19}\text{ C})(442\text{ kV})/(1.67 \times 10^{-27}\text{ kg})]^{1/2} = 9.21$ Mm/s.

Problem

24. A sphere of radius R carries a negative charge of magnitude Q, distributed in a spherically symmetric way. Find the "escape speed" for a proton at the sphere's surface—that is, the speed that would enable the proton to escape to arbitrarily large distances.

Solution

The work done by the electric field, when a proton escapes from the surface to an infinite distance, equals the change in kinetic energy, or $-e(V_\infty - V_{\text{surf}}) = eV_{\text{surf}} = K_\infty - K_{\text{surf}} = -\frac{1}{2}mv^2$. (We assumed zero kinetic energy for the proton at infinity, and that the sphere is stationary.) For a uniformly negatively charged sphere, $V_{\text{surf}} = -kQ/R$, so $v = \sqrt{2keQ/mR}$.

Problem

25. A thin spherical shell of charge has radius R and total charge Q distributed uniformly over its surface. What is the potential at its center?

Solution

The potential at the surface of the shell is kQ/R (as in Example 25-3). The electric field inside a uniformly charged shell is zero, so the potential anywhere inside is a constant, equal, therefore, to its value at the surface.

Problem

26. A solid sphere of radius R carries a net charge Q distributed uniformly throughout its volume. Find the potential difference from the sphere's surface to its center. *Hint:* Consult Example 24-1.

Solution

The electric field inside a uniformly charged sphere is radially symmetric with strength $E = kQr/R^3$. Then $V(R) - V(0) = -\int_0^R (kQr/R^3)\,dr = -kQ/2R$. (The potential is higher at the center if Q is positive.)

Problem

27. Find the potential as a function of position in an electric field given by $\mathbf{E} = ax\,\hat{\mathbf{i}}$, where a is a constant and where $V = 0$ at $x = 0$.

Solution

Since $V(0) = 0$, $V(\mathbf{r}) = -\int_0^r \mathbf{E}\cdot d\mathbf{r} = -\int_0^r ax\,\hat{\mathbf{i}}\cdot d\mathbf{r} = -\int_0^x ax\,dx = -\frac{1}{2}ax^2$.

Problem

29. The potential difference between the surface of a 3.0-cm-diameter power line and a point 1.0 m distant is 3.9 kV. What is the line charge density on the power line?

Solution

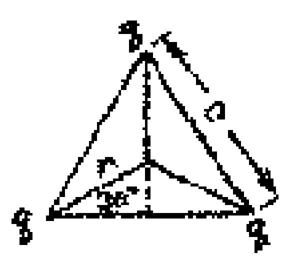
If we approximate the potential from the line by that from an infinitely long charged wire, Equation 25-5 can be used to find λ: $\lambda = 2\pi\varepsilon_0\,\Delta V_{AB}/\ln(r_A/r_B) = (3.9\text{ kV})[2(9\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\ln(100/1.5)]^{-1} = 51.6\text{ nC/m}$. (Note: $\Delta V_{AB} = V_B - V_A$ so B is at the surface of the wire and A is 100 cm distant.)

Problem

30. Three equal charges q form an equilateral triangle of side a. Find the potential at the center of the triangle.

Solution

The center is equidistant from each vertex, and $r = a/2\cos 30° = a/\sqrt{3}$. Each charge contributes equally to the potential, so $V = 3kq/r = 3\sqrt{3}kq/a$.



Problem 30 Solution.

Problem

32. Two identical charges q lie on the x-axis at $\pm a$. (a) Find an expression for the potential at all points in the x-y plane. (b) Show that your result reduces to the potential of a point charge for distances large compared with a.

Solution

(a) Equation 25-6 and some geometry give $V(\mathbf{r}) = \Sigma kq_i/r_i = kq[|\mathbf{r} - a\hat{\mathbf{i}}|^{-1} + |\mathbf{r} + a\hat{\mathbf{i}}|^{-1}] = kq\{[(x - a)^2 + y^2]^{-1/2} + [(x + a)^2 + y^2]^{-1/2}\}$. (b) If $r^2 = x^2 + y^2 \gg a$, a can be neglected relative to x or y, so $V(r) \to 2kq/r$, which is the potential of a point charge of magnitude $2q$.

Problem

37. A thin ring of radius R carries a charge $3Q$ distributed uniformly over three-fourths of its circumference, and $-Q$ over the rest. What is the potential at the center of the ring?

Solution

The result in Example 25-6 did not depend on the ring being uniformly charged. For a point on the axis of the ring, the geometrical factors are the same, and $\int_{\text{ring}} dq = Q_{\text{tot}}$ for any arbitrary charge distribution, so $V = kQ_{\text{tot}}(x^2 + a^2)^{-1/2}$ still holds. Thus, at the center $(x = 0)$ of a ring of total charge $Q_{\text{tot}} = 3Q - Q = 2Q$, and radius $a = R$, the potential is $V = 2kQ/R$.

Problem

39. The annulus shown in Fig. 25-40 carries a uniform surface charge density σ. Find an expression for the potential at an arbitrary point P on its axis.

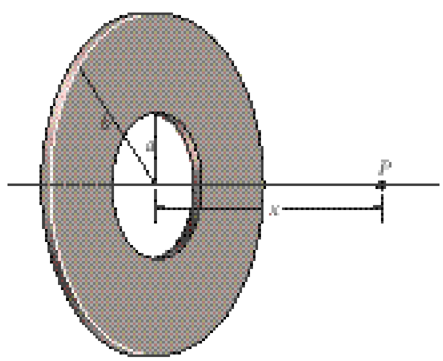


FIGURE 25-40 Problem 39.

Solution

The annulus can be considered to be composed of thin rings of radius $r\ (a \le r \le b)$ and charge $dq = 2\pi\sigma r\,dr$ (see Example 25-7 and Fig.'s 25-15 and 16). The element of potential from a ring on its axis, a distance x from the center, is $dV = k\,dq/\sqrt{x^2 + r^2}$ (see Example 25-6) so the potential from the whole annulus is:

$$V = \int dV = 2\pi\sigma k \int_a^b \frac{r\,dr}{\sqrt{x^2 + r^2}} = 2\pi k\sigma \left|\sqrt{x^2 + r^2}\right|_a^b = 2\pi k\sigma\left(\sqrt{x^2 + b^2} - \sqrt{x^2 + a^2}\right).$$

(Note: This reduces to the potential on the axis of a uniformly charged disk if $a \to 0$.)

Problem

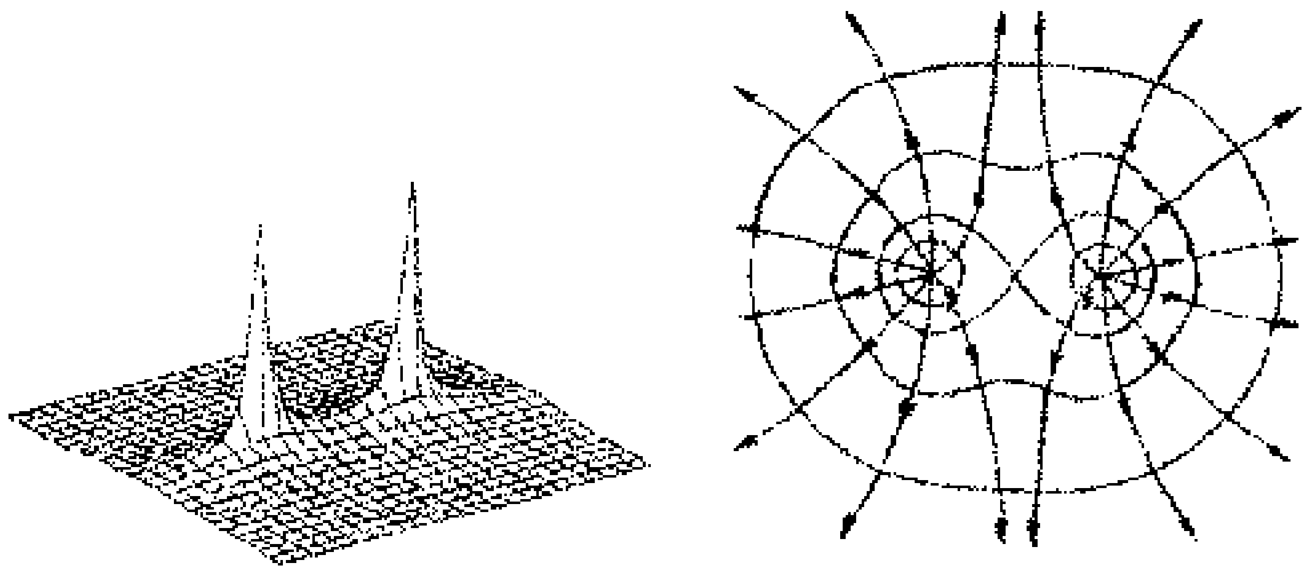
41. (a) Find the potential as a function of position in the electric field $\mathbf{E} = E_0(\hat{\mathbf{i}} + \hat{\mathbf{j}})$, where $E_0 = 150$ V/m. Take the zero of potential at the origin. (b) Find the potential difference from the point $x = 2.0$ m, $y = 1.0$ m to the point $x = 3.5$ m, $y = -1.5$ m.

Solution

(a) Equation 25-2b gives the potential for a uniform field. Take the zero of potential at the origin (point A in Equation 25-2b) and let $\mathbf{l} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ be the vector from the origin to the field point (point B in Equation 25-2b). Then $\Delta V_{AB} = V_B - V_A = V(\mathbf{r}) - 0 = V(x, y) = -E_0(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cdot \mathbf{r} = -E_0(x + y)$. (The potential is independent of z, so we wrote $V(\mathbf{r}) = V(x, y)$.) (b) $V(3.5\text{ m}, -1.5\text{ m}) - V(2.0\text{ m}, 1.0\text{ m}) = -(150\text{ V/m})(3.5\text{ m} - 1.5\text{ m} - 2.0\text{ m} - 1.0\text{ m}) = 150\text{ V}$.

Problem

46. Sketch some equipotentials and field lines for a distribution consisting of two equal point charges.



Problem 46 Solution.

Solution

The equipotential surfaces for two equal point charges, q, located on the x-axis at $\pm a$, are given by

$$\left(\frac{4\pi\varepsilon_0}{q}\right) V(x, y, z) = \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} = \text{constant}$$

Lines of force are perpendicular to the equipotentials at every point. We sketched the field in the x-y plane, without adhering to any consistent numerical mapping convention, but in sufficient detail to display its general shape, in the vicinity of the charges. Standard calculus techniques and a personal computer help in preparing such pictures. A three-dimensional plot, as in Fig. 25-14, has also been included; caveat—the lines on the three-dimensional plot are *not* equipotentials.

Problem

49. Use the result of Example 25-6 to determine the on-axis field of a charged ring, and verify that your answer agrees with the result of Example 23-8.

Solution

On the axis of a uniformly charged ring (the x-axis), $V = kQ/\sqrt{x^2 + a^2}$ (Equation 25-9), and the electric field only has an x component (by symmetry). Then $E = (-dV/dx)\hat{\mathbf{i}} = kQx(x^2 + a^2)^{-3/2}\hat{\mathbf{i}}$, in accord with Example 23-8. (In general, one needs to know the potential in a 3-dimensional region in order to calculate the field from its partial derivatives.)

Problem

52. (a) What is the maximum potential (measured from infinity) for the sphere of Example 25-3 before dielectric breakdown of air occurs at the sphere's surface? (Breakdown of air occurs at a field strength of 3 MV/m.) (b) What is the charge on the sphere when it's at this potential?

Solution

(b) Dielectric breakdown in the air occurs if the field at the surface, $E = \sigma/\varepsilon_0$, exceeds 3×10^6 V/m. Therefore, the charge (for a uniformly charged sphere) must not be greater than $q = 4\pi R^2\sigma = 4\pi\varepsilon_0 ER^2 = (3 \times 10^6 \text{ V/m})(2.3 \text{ m})^2 \div (9 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2) = 1.76$ mC. (a) From Equation 25-11, $V = kq/R = RE = 6.9$ MV.

Problem

54. A large metal sphere has three times the diameter of a smaller sphere and carries three times as much charge. Both spheres are isolated, so their surface charge densities are uniform. Compare (a) the potentials and (b) the electric field strengths at their surfaces.

Solution

(a) The potential of an isolated metal sphere, with charge Q and radius R, is kQ/R, so a sphere with charge $3Q$ and radius $3R$ has the same potential. (b) However, the electric field at the surface of the smaller sphere is $\sigma/\varepsilon_0 = kQ/R^2$, so tripling Q and R reduces the surface field by a factor of $\frac{1}{3}$.

Problem

55. Two metal spheres each 1.0 cm in radius are far apart. One sphere carries 38 nC of charge, the other -10 nC. (a) What is the potential on each? (b) If the spheres are connected by a thin wire, what will be the potential on each once equilibrium is reached? (c) How much charge must move between the spheres in order to achieve equilibrium?

Solution

(a) Since the spheres are far apart (approximately isolated), we can use Equation 25-11 to find their potentials: $V_1 = kQ_1/R_1 = (9 \text{ GN}\cdot\text{m}^2/\text{C}^2)(38 \text{ nC})/(1 \text{ cm}) = 34.2$ kV and $V_2 = kQ_2/R_2 = -9$ kV. (b) When connected by a thin wire, the spheres reach electrostatic equilibrium with the same potential, so $V = kQ_1'/R_1 = kQ_2'/R_2$. Since the radii are equal, so must be the charges, $Q_1' = Q_2'$. The total charge is $38 \text{ nC} - 10 \text{ nC} = 28 \text{ nC} = Q_1' + Q_2' = 2Q_1'$ (if we assume that the wire is so thin that it has a negligible charge), so $Q_1' = Q_2' = 14$ nC. Then $V' = k(14 \text{ nC})/(1 \text{ cm}) = 12.6$ kV. (c) In this process, the first sphere loses $38 - 14 = 24$ nC to the second.