(1)   In QM, every observable quantity (position, momentum, energy, etc.) is represented by an *operator*. This operator operates on the wave function. Thus the position x, Potential Energy U(x) or any function of x operates by simply *multiplying* the wave function by that function: e.g. $x\psi(x); U(x)\psi(x); x^2\psi(x)$ etc.

On the other hand, momentum p is represented by the differential operator $-i\hbar\dfrac{d}{dx}$ operating on ψ(x) (We are not writing this as a partial differential as we are considering only the space-dependent part of the full wave function ψ(x,t) and only one dimension of space, i.e. along the x-axis)

so

$$p = -i\hbar\frac{d}{dx}$$

$$p^2 = \left[-i\hbar\frac{d}{dx}\right]\left[-i\hbar\frac{d}{dx}\right] = -\hbar^2\frac{d^2}{dx^2}$$

$$\text{K.E. operator} = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\text{Total Energy operator} = \text{K.E.} + \text{P.E.} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)$$

This is usually called the *Hamiltonian* operator and is denoted by H.

   (2) The *average* value (called the *expectation* value) for a series of measurements of the value of some observable quantity on a QM particle or system is given by: $\int\limits_{-\infty}^{\infty} dx\psi^*(x)O\psi(x)$ (provided $\int\limits_{-\infty}^{\infty} dxP(x)f(x)$ is properly normalized)

[This is the operator analog of the result in statistics that the expectation value of some function of x , f(x) is given by $\int\limits_{-\infty}^{\infty} dxP(x)f(x)$ where P(x)dx is the probability distribution for getting a value of x between x and x + dx.]

PRACTICE EVALUATING SOME OF THESE EXPECTATION VALUES FOR SOME OF THE KNOWN WAVE FUNCTIONS YOU ARE FAMILIAR WITH. (These may involve evaluating messy integrals!)

(3) If an operator operating on ψ(x) gives simply a NUMBER (not a function of x) times that ψ(x), then ψ(x) is called an *eigenfunction* of the operator and the number that appears in front is called an *eigenvalue* (In general, to every unique eigenfunction of an operator corresponds a particular eigenvalue of that operator).A particle with this wave function is said to be in a particular *eigenstate* of this operator.

Thus consider the wave function $Ae^{i(kx-\omega t)}$ It is easily seen that p operating on

$Ae^{i(kx-\omega t)}$ yields $\hbar k Ae^{i(kx-\omega t)}$ showing that this function is an eigenfunction of the momentum operator with eigenvalue $\hbar k$

From what is given above, we can also see that the expectation value of p

$\langle p \rangle = \int_{-\infty}^{\infty} dx \psi^*(x,t) p \psi(x,t) = \hbar k$, pulling out the constant $\hbar k$, since if the wave

function is properly normalized, $\int_{-\infty}^{\infty} dx \psi^*(x,t) \psi(x,t) = 1$

Similarly, $\langle p^2 \rangle = \int_{-\infty}^{\infty} dx \psi^*(x,t) p^2 \psi(x,t) = (\hbar k)^2$ and so on…..

the above results imply that for this function $\langle p^2 \rangle - \langle p \rangle^2 = 0$, i.e. Δp = 0
(There is no conflict with the Heisenberg uncertainty principle here, since this wave function has $\psi^*(x,t)\psi(x,t) = cons \tan t$ and so Δx is infinite)
SO,…
if a particle is in a particular eigenstate of an operator O , the expectation value of that operator O is CERTAIN to be the eigenvalue corresponding to that state, with NO UNCERTAINTY.
Note: A sin (kx) is NOT an eigenfunction of p, but IS an eigenfunction of p^2 so it is not an eigenfunction of the momentum operator, but is an eigenfunction of the Kinetic Energy operator. NONE of these functions is an eigenfunction of the position operator x. CHECK THESE FACTS FOR YOURSELF.

From what has been said, going back to the definition above of the Hamiltonian Operator (or Total Energy operator) you can see that ANY solution of the stationary Schrodinger Equation must be an eigenfunction of the Hamiltonian operator, with E (the solved energy for that state) as the corresponding eigenvalue. (VERIFY THAT THIS IS TRUE)

In general the S.E. will have an infinite number of solutions with corresponding E's and these correspond to the complete set of

eigenfunctions (wave functions) and eigenvalues (energy levels) of the Hamiltonian Operator. The Hamiltonian operator plays a special role in QM, and the game is usually reduced to finding the eigenfunctions and eigenvalues of this operator for various forms of the potential U(x).

If you understand this, you have gone a long way to understanding the mathematical machinery of Quantum Mechanics!!