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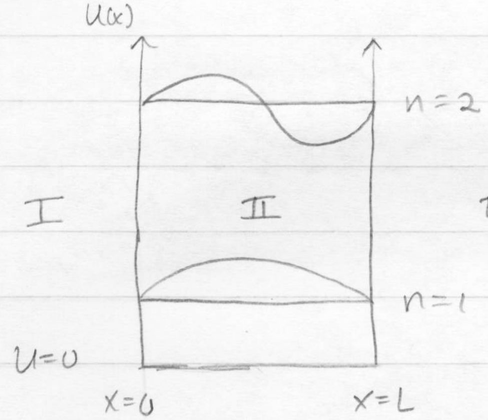

For regions I and III, $U=\infty$ so $\psi(x)=0$.

For region II the TISE is

$$\frac{d^2}{dx^2}\psi(x) + \frac{2mE}{\hbar^2}\psi(x) = 0$$

which has the general solution

$$\psi(x) = Ae^{ik_2x} + Be^{-ik_2x} \quad \text{where } k_2 = \frac{\sqrt{2mE}}{\hbar}$$

The continuity of ψ requires $\psi(0)=\psi(L)=0$.
Applying these boundary conditions gives
$A+B=0$ and $Ae^{ik_2L}+Be^{-ik_2L}=0$. Combining these,
$A(e^{ik_2L}-e^{-ik_2L})=0$ and using the identity $e^{ix}=\cos(x)+i\sin(x)$
we obtain $2iA\sin(k_2L)=0$. Since $2iA$ is not zero,
we require $\sin(k_2L)=0$ so $k_2L=n\pi$ $(n=1,2,3\ldots)$.
Therefore, $\boxed{k_2^{(n)}=\frac{n\pi}{L}}$. Going back to our general solution,

we now have $\psi(x)=2iA\sin(k_2x)=A'\sin(k_2x)$
where I have simply redefined the normalization constant.
Now we need to normalize $\psi(x)$ to find A'.

$$\int_{-\infty}^{\infty}|\psi|^2dx = \int_0^L (A')^2\sin^2(k_2^{(n)}x)\,dx = 1$$

You can do this integral by using $\sin^2(u)=\frac{1}{2}-\frac{1}{2}\cos(2u)$
to obtain $A'=\sqrt{\frac{2}{L}}$, so we have $\boxed{\psi_n(x)=\sqrt{\frac{2}{L}}\sin(k_2^{(n)}x)}$

Plugging our solution into the TISE, we obtain:

$$\frac{d\psi}{dx} = k_2^{(n)}\sqrt{\frac{2}{L}}\cos\left(k_2^{(n)}x\right), \quad \frac{d^2\psi}{dx^2} = -\left(k_2^{(n)}\right)^2\sqrt{\frac{2}{L}}\sin\left(k_2^{(n)}x\right)$$

$$-\frac{\hbar^2}{2m}\left(-\left(k_2^{(n)}\right)^2\psi(x)\right) + (0)\psi(x) = E\psi(x)$$

$$\therefore \boxed{E = \frac{\hbar^2\left(k_2^{(n)}\right)^2}{2m} = \frac{\hbar^2 n^2\pi^2}{2mL^2}}$$

In particular, for $n=1$ and $n=2$, we have

$$\boxed{\begin{array}{ll} \psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right) & E_1 = \frac{\hbar^2\pi^2}{2mL^2} \\ \psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) & E_2 = \frac{4\hbar^2\pi^2}{2mL^2} = \frac{2\hbar^2\pi^2}{mL^2} \end{array}}$$

These wavefunctions are sketched on the previous page.

A maximum occurs when $\frac{d}{dx}\left(\psi^2(x)\right) = 0$ and $\frac{d^2}{dx^2}\left(\psi^2(x)\right) < 0$

A minimum occurs when $\frac{d}{dx}\left(\psi^2(x)\right) = 0$ and $\frac{d^2}{dx^2}\left(\psi^2(x)\right) > 0$.

$$\frac{d}{dx}\left(\psi^2(x)\right) = \frac{2}{L}\frac{d}{dx}\left(\sin^2(kx)\right) = \frac{4k}{L}\sin(kx)\cos(kx) = 0$$

Which is zero when $\sin(kx) = 0$ or $\cos(kx) = 0$

$\sin(kx) = 0 \Rightarrow \frac{n\pi}{L}x = m\pi \quad (m = 0, 1, 2, \ldots)$

$\cos(kx) = 0 \Rightarrow \frac{n\pi}{L}x = \frac{p\pi}{2} \quad (p = 1, 3, 5, \ldots)$

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For $n=1$, these conditions become

$$\frac{x}{L} = m \quad (m=0,1,2,\ldots)$$

$$\frac{x}{L} = \frac{p}{2} \quad (p=1,3,5,\ldots)$$

but only $m=0,1$ and $p=1$ give x in the range $0 \le x \le L$.

$$\frac{d^2}{dx^2}\left(\psi^2(x)\right) = \frac{d}{dx}\left(\frac{4k}{L}\sin(kx)\cos(kx)\right) = \frac{4k^2}{L}\left(\cos^2(kx) - \sin^2(kx)\right)$$

For $m=0$, $\boxed{x=0}$ and $\frac{d^2}{dx^2}\left(\psi^2(x)\right) = +\frac{4k^2}{L} > 0 \Rightarrow \boxed{\text{minimum}}$

For $m=1$, $\boxed{x=L}$ and $\frac{d^2}{dx^2}\left(\psi^2(x)\right) = +\frac{4k^2}{L} > 0 \Rightarrow \boxed{\text{minimum}}$

For $p=1$, $\boxed{x=L/2}$ and $\frac{d^2}{dx^2}\left(\psi^2(x)\right) = -\frac{4k^2}{L} < 0 \Rightarrow \boxed{\text{maximum}}$

For $n=2$,

$$\frac{2x}{L} = m \quad (m=0,1,2,\ldots)$$

$$\frac{2x}{L} = \frac{p}{2} \quad (p=1,3,5\ldots)$$

$$\left.\boxed{\begin{array}{l} m=0 \Rightarrow x=0, \text{ minimum} \\ m=1 \Rightarrow x=L/2, \text{ minimum} \\ m=2 \Rightarrow x=L, \text{ minimum} \\ p=1 \Rightarrow x=L/4, \text{ maximum} \\ p=3 \Rightarrow x=3L/4, \text{ maximum} \end{array}}\right\}$$

Same reasoning as above but note that k changes when going from $n=1$ to $n=2$.

All minima identified here correspond to $\psi(x)=0 \Rightarrow |\psi|^2=0$

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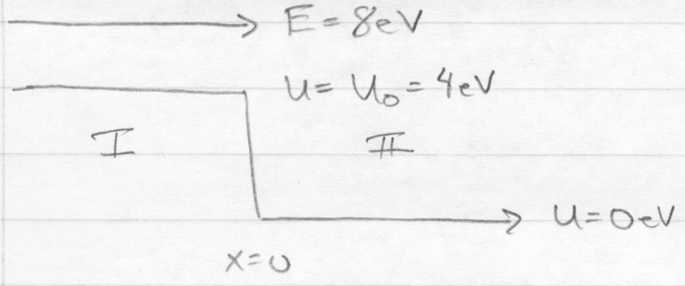

TISE: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + U(x)\psi(x) = E\psi(x)$

In our case $U = U_0$ (region I) or $U = 0$ (region II) so

$$\frac{d^2}{dx^2}\psi(x) + \underbrace{\frac{2m}{\hbar^2}(E-U)}_{k^2}\psi(x) = 0 \Rightarrow \psi(x) = Ae^{ikx} + Be^{-ikx}$$

Note that $E > U$ in both region I and II so k is real and we have an oscillating solution everywhere.

Assume that the right moving wave in region I has amplitude A, the left moving wave in region I has amplitude B and the right moving (and only) wave in region II has amplitude C.

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$$
$$\psi_{II}(x) = Ce^{ik_2x}$$

$$\boxed{\begin{aligned} k_1 &= \sqrt{2m(E-U_0)}/\hbar \\ k_2 &= \sqrt{2mE}/\hbar \end{aligned}}$$

ψ must be continuous and $\frac{d\psi}{dx}$ must be continuous:

$A + B = C$ ①

$ik_1(A-B) = ik_2C$

$A - B = \frac{k_2}{k_1}C$ ②

$$① + ② \Rightarrow 2A = \left(1+\frac{k_2}{k_1}\right)C \Rightarrow \frac{C}{A} = \frac{2}{\left(1+\frac{k_2}{k_1}\right)} \Rightarrow \boxed{\left|\frac{C}{A}\right|^2 = \frac{4k_1^2}{(k_1+k_2)^2}} = T$$

$$① - ② \Rightarrow 2B = \left(1-\frac{k_2}{k_1}\right)C = \frac{\left(1-\frac{k_2}{k_1}\right)}{\left(1+k_2/k_1\right)}2A \Rightarrow \boxed{\left|\frac{B}{A}\right|^2 = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 = R}$$

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$$K_1 = \sqrt{2(9.1\times10^{-31}\,kg)(4eV \times 1.6\times10^{-19}\,J/eV)}\,/\,1.055\times10^{-34}\,Js$$

$$= 1.023\times10^{10}\,m^{-1}$$

$$K_2 = \sqrt{2(9.1\times10^{-31}\,kg)(8eV\times1.6\times10^{-19}\,J/eV)}\,/\,1.055\times10^{-34}\,Js$$

$$= 1.447\times10^{10}\,m^{-1}$$

$$T = \frac{4K_1^2}{(K_1+K_2)^2} = \boxed{6.86\times10^{-1}}$$

$$R = \left(\frac{K_1-K_2}{K_1+K_2}\right)^2 = \boxed{2.95\times10^{-2}}$$

Note that $R+T \neq 1$ but $\boxed{R + \frac{K_2}{K_1}T = 1}$

so the current is conserved.