**PHYSICS 140A : STATISTICAL PHYSICS**
**FINAL EXAMINATION**
**(do all four problems)**

(1) The entropy for a peculiar thermodynamic system has the form

$$S(E,V,N) = Nk_B \left\{ \left(\frac{E}{N\varepsilon_0}\right)^{1/3} + \left(\frac{V}{Nv_0}\right)^{1/2} \right\} ,$$

where $\varepsilon_0$ and $v_0$ are constants with dimensions of energy and volume, respectively.

(a) Find the equation of state $p = p(T,V,N)$.
[5 points]

(b) Find the work done along an isotherm in the $(V,p)$ plane between points A and B in terms of the temperature $T$, the number of particles $N$, and the pressures $p_A$ and $p_B$.
[10 points]

(c) Find $\mu(T,p)$.
[10 points]

(2) Consider a set of $N$ noninteracting crystalline defects characterized by a dipole moment $\boldsymbol{p} = p_0\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ can point in any of six directions: $\pm\hat{\boldsymbol{x}}$, $\pm\hat{\boldsymbol{y}}$, and $\pm\hat{\boldsymbol{z}}$. In the absence of an external field, the energies for these configurations are $\varepsilon(\pm\hat{\boldsymbol{x}}) = \varepsilon(\pm\hat{\boldsymbol{y}}) = \varepsilon_0$ and $\varepsilon(\pm\hat{\boldsymbol{z}}) = 0$.

(a) Find the free energy $F(T,N)$.
[10 points]

(b) Now let there be an external electric field $\boldsymbol{E} = E\,\hat{\boldsymbol{z}}$. The energy in the presence of the field is augmented by $\Delta\varepsilon = -\boldsymbol{p}\cdot\boldsymbol{E}$. Compute the total dipole moment $\boldsymbol{P} = \sum_i \langle \boldsymbol{p}_i \rangle$.
[5 points]

(c) Compute the electric susceptibility $\chi_E^{zz} = \frac{1}{V}\frac{\partial P_z}{\partial E_z}$ at $\boldsymbol{E} = 0$.
[5 points]

(d) Find an expression for the entropy $S(T,N,E)$ when $\varepsilon_0 = 0$.
[5 points]

(3) A bosonic gas is known to have a power law density of states $g(\varepsilon) = A\,\varepsilon^{\sigma}$ per unit volume, where $\sigma$ is a real number.

(a) Experimentalists measure $T_c$ as a function of the number density $n$ and make a log-log plot of their results. They find a beautiful straight line with slope $\frac{3}{7}$. That is, $T_c(n) \propto n^{3/7}$. Assuming the phase transition they observe is an ideal Bose-Einstein condensation, find the value of $\sigma$.
`[5 points]`

(b) For $T < T_c$, find the heat capacity $C_V$.
`[5 points]`

(c) For $T > T_c$, find an expression for $p(T, z)$, where $z = e^{\beta\mu}$ is the fugacity. Recall the definition of the polylogarithm (or generalized Riemann zeta function)[1],

$$\mathrm{Li}_q(z) \equiv \frac{1}{\Gamma(q)} \int_0^\infty \! dt \, \frac{t^{q-1}}{z^{-1} e^t - 1} = \sum_{n=1}^\infty \frac{z^n}{n^q} \, ,$$

where $\Gamma(q) = \int_0^\infty dt\, t^{q-1}\, e^{-t}$ is the Gamma function.
`[5 points]`

(d) If these particles were fermions rather than bosons, find (i) the Fermi energy $\varepsilon_F(n)$ and (ii) the pressure $p(n)$ as functions of the density $n$ at $T = 0$.
`[10 points]`

(4) Provide brief but substantial answers to the following:

(a) Consider a three-dimensional gas of $N$ classical particles of mass $m$ in a uniform gravitational field $g$. Assume $z \geq 0$ and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. Find the heat capacity $C_V$.
`[7 points]`

(b) Consider a system with a single phase space coordinate $\phi$ which lives on a circle. Now consider three dynamical systems on this phase space:

$$\text{(i)}\ \dot{\phi} = 0 \quad , \quad \text{(ii)}\ \dot{\phi} = 1 \quad , \quad \text{(iii)}\ \dot{\phi} = 2 - \cos\phi \, .$$

For each of these systems, tell whether it is recurrent, ergodic, both, or neither, and explain your reasoning.
`[6 points]`

(c) Explain Boltzmann's $H$-theorem.
`[6 points]`

[1] In the notes and in class we used the notation $\zeta_q(z)$ for the polylogarithm, but for those of you who have yet to master the scribal complexities of the Greek $\zeta$, you can use the notation $\mathrm{Li}_q(z)$ instead.

(d) $\nu$ moles of gaseous Argon at an initial temperature $T_{\mathrm{A}}$ and volume $V_{\mathrm{A}} = 1.0\,\mathrm{L}$ undergo an adiabatic free expansion to an intermediate state of volume $V_{\mathrm{B}} = 2.0\,\mathrm{L}$. After coming to equilibrium, this process is followed by a reversible adiabatic expansion to a final state of volume $V_{\mathrm{C}} = 3.0\,\mathrm{L}$. Let $S_{\mathrm{A}}$ denote the initial entropy of the gas. Find the temperatures $T_{\mathrm{B,C}}$ and the entropies $S_{\mathrm{B,C}}$. Then repeat the calculation assuming the first expansion (from A to B) is a reversible adiabatic expansion and the second (from B to C) an adiabatic free expansion.
[6 points]

(5) Match the Jonathan Coulton song lines in the left column with their following lines in the right column.
[30 quatloos extra credit]

| | |
|---|---|
| (a) That was a joke – haha – fat chance | (1) I can see the day unfold in front of me |
| (b) Saw a vision in his head | (2) I'm glad to see you take constructive criticism well |
| (c) I try to medicate my concentration haze | (3) And this mountain is covered with wolves |
| (d) I've been patient, I've been gracious | (4) A bulbous pointy form |
| (e) I guess we'll table this for now | (5) Hearing the whirr of the servos inside |
| (f) She'll eye me suspiciously | (6) Anyway this cake is great |