PHYSICS 140A : STATISTICAL PHYSICS
FINAL EXAMINATION
(do all four problems)

(1) The entropy for a peculiar thermodynamic system has the form

$$S(E,V,N) = Nk_{\mathrm{B}}\left\{\left(\frac{E}{N\varepsilon_0}\right)^{1/3} + \left(\frac{V}{Nv_0}\right)^{1/2}\right\} ,$$

where ε_0 and v_0 are constants with dimensions of energy and volume, respectively.

(a) Find the equation of state $p = p(T,V,N)$.
[5 points]

(b) Find the work done along an isotherm in the (V,p) plane between points A and B in terms of the temperature T, the number of particles N, and the pressures p_{A} and p_{B}.
[10 points]

(c) Find $\mu(T,p)$.
[10 points]

Solution :

(a) (a) We have

$$p = T\left(\frac{\partial S}{\partial V}\right)_{E,N} = \frac{k_{\mathrm{B}}T}{2v_0}\left(\frac{V}{Nv_0}\right)^{-1/2} .$$

(b) We use the result of part (a) to obtain

$$W_{\mathrm{AB}} = \int\limits_{\mathrm{A}}^{\mathrm{B}} p\,dV = Nk_{\mathrm{B}}T\left(\frac{V}{Nv_0}\right)^{1/2}\Bigg|_{\mathrm{A}}^{\mathrm{B}} = \frac{N(k_{\mathrm{B}}T)^2}{2v_0}\left(\frac{1}{p_{\mathrm{B}}} - \frac{1}{p_{\mathrm{A}}}\right) .$$

(c) We have

$$\mu = T\left(\frac{\partial S}{\partial N}\right)_{E,V} = \tfrac{2}{3}k_{\mathrm{B}}T\left(\frac{E}{N\varepsilon_0}\right)^{1/3} + \tfrac{1}{2}k_{\mathrm{B}}T\left(\frac{V}{Nv_0}\right)^{1/2} .$$

The temperature is given by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{k_{\mathrm{B}}}{3\varepsilon_0}\left(\frac{E}{N\varepsilon_0}\right)^{-2/3} .$$

Thus, using

$$\frac{E}{N\varepsilon_0} = \left(\frac{k_{\mathrm{B}}T}{3\varepsilon_0}\right)^{3/2} \qquad , \qquad \frac{V}{Nv_0} = \left(\frac{k_{\mathrm{B}}T}{2p\,v_0}\right)^2 ,$$

we obtain

$$\mu(T,p) = \frac{2(k_\text{B}T)^{3/2}}{3\sqrt{3}\,\varepsilon_0^{1/2}} + \frac{(k_\text{B}T)^2}{4pv_0} \,.$$

(2) Consider a set of N noninteracting crystalline defects characterized by a dipole moment $\boldsymbol{p} = p_0\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ can point in any of six directions: $\pm\hat{\boldsymbol{x}}$, $\pm\hat{\boldsymbol{y}}$, and $\pm\hat{\boldsymbol{z}}$. In the absence of an external field, the energies for these configurations are $\varepsilon(\pm\hat{\boldsymbol{x}}) = \varepsilon(\pm\hat{\boldsymbol{y}}) = \varepsilon_0$ and $\varepsilon(\pm\hat{\boldsymbol{z}}) = 0$.

(a) Find the free energy $F(T,N)$.
`[10 points]`

(b) Now let there be an external electric field $\boldsymbol{E} = E\,\hat{\boldsymbol{z}}$. The energy in the presence of the field is augmented by $\Delta\varepsilon = -\boldsymbol{p}\cdot\boldsymbol{E}$. Compute the total dipole moment $\boldsymbol{P} = \sum_i \langle \boldsymbol{p}_i \rangle$.
`[5 points]`

(c) Compute the electric susceptibility $\chi_E^{zz} = \frac{1}{V}\frac{\partial P_z}{\partial E_z}$ at $\boldsymbol{E} = 0$.
`[5 points]`

(d) Find an expression for the entropy $S(T,N,E)$ when $\varepsilon_0 = 0$.
`[5 points]`

Solution :

(a) We have $Z = \xi^N$ where the single particle partition function is

$$\xi = \mathsf{Tr}\, e^{-\beta h} = 4\, e^{-\beta\varepsilon_0} + 2 \,.$$

Thus,

$$F(T,N) = -k_\text{B}T \ln Z = -Nk_\text{B}T\, \ln\Big(2 + 4\, e^{-\varepsilon_0/k_\text{B}T}\Big) \,.$$

(b) Including effects of the electric field, we have

$$F(T,N) = -k_\text{B}T \ln Z = -Nk_\text{B}T\, \ln\Big(2\cosh\Big(\tfrac{p_0E}{k_\text{B}T}\Big) + 4\, e^{-\varepsilon_0/k_\text{B}T}\Big) \,.$$

The electric polarization is clearly aligned along $\hat{\boldsymbol{z}}$, *i.e.* $\boldsymbol{P} = P(T,N,E)\,\hat{\boldsymbol{z}}$, with

$$P = -\left(\frac{\partial F}{\partial E}\right)_{T,N} = \frac{Np_0\,\sinh(p_0E/k_\text{B}T)}{2\,e^{-\varepsilon_0/k_\text{B}T} + \cosh(p_0E/k_\text{B}T)} \,.$$

(c) We expand P to linear order in E and differentiate, yielding

$$\chi_E^{zz} = \frac{N}{V}\cdot\frac{1}{2\,e^{-\varepsilon_0/k_\text{B}T} + 1}\cdot\frac{p_0^2}{k_\text{B}T} \,.$$

(d) Setting $\varepsilon_0 = 0$, we have

$$F(T,N) = -Nk_{\rm B}T\,\ln\Big(4 + 2\cosh\big(p_0\,E/k_{\rm B}T\big)\Big)\ .$$

The entropy is then

$$S = -\bigg(\frac{\partial F}{\partial T}\bigg)_{\!N} = Nk_{\rm B}\left[\ln\Big(4 + 2\cosh\big(p_0 E/k_{\rm B}T\big)\Big) - \frac{(p_0 E/k_{\rm B}T)\sinh\big(p_0 E/k_{\rm B}T\big)}{2 + \cosh\big(p_0 E/k_{\rm B}T\big)}\right] .$$

(3) A bosonic gas is known to have a power law density of states $g(\varepsilon) = A\,\varepsilon^\sigma$ per unit volume, where σ is a real number.

(a) Experimentalists measure $T_{\rm c}$ as a function of the number density n and make a log-log plot of their results. They find a beautiful straight line with slope $\frac{3}{7}$. That is, $T_{\rm c}(n) \propto n^{3/7}$. Assuming the phase transition they observe is an ideal Bose-Einstein condensation, find the value of σ.
[5 points]

(b) For $T < T_{\rm c}$, find the heat capacity C_V.
[5 points]

(c) For $T > T_{\rm c}$, find an expression for $p(T,z)$, where $z = e^{\beta\mu}$ is the fugacity. Recall the definition of the polylogarithm (or generalized Riemann zeta function)[1],

$$\mathrm{Li}_q(z) \equiv \frac{1}{\Gamma(q)}\int\limits_0^\infty\!dt\,\frac{t^{q-1}}{z^{-1}e^t - 1} = \sum_{n=1}^\infty \frac{z^n}{n^q}\ ,$$

where $\Gamma(q) = \int\limits_0^\infty\! dt\; t^{q-1}\,e^{-t}$ is the Gamma function.
[5 points]

(d) If these particles were fermions rather than bosons, find (i) the Fermi energy $\varepsilon_{\rm F}(n)$ and (ii) the pressure $p(n)$ as functions of the density n at $T = 0$.
[10 points]

Solution :

(a) At $T = T_{\rm c}$, we have $\mu = 0$ and $n_0 = 0$, hence

$$n = \int\limits_{-\infty}^\infty\! d\varepsilon\,\frac{g(\varepsilon)}{e^{\varepsilon/k_{\rm B}T_{\rm c}} - 1} = \Gamma(1+\sigma)\,\zeta(1+\sigma)\,A\,(k_{\rm B}T_{\rm c})^{1+\sigma}\ .$$

[1] In the notes and in class we used the notation $\zeta_q(z)$ for the polylogarithm, but for those of you who have yet to master the scribal complexities of the Greek ζ, you can use the notation $\mathrm{Li}_q(z)$ instead.

Thus, $T_c \propto n^{\frac{1}{1+\sigma}} = n^{3/7}$ which means $\sigma = \frac{4}{3}$.

(b) For $T < T_c$ we have $\mu = 0$, but the condensate carries no energy. Thus,

$$E = V \int_{-\infty}^{\infty} d\varepsilon \, \frac{\varepsilon \, g(\varepsilon)}{e^{\varepsilon/k_B T} - 1} = \Gamma(2+\sigma)\, \zeta(2+\sigma)\, A\, (k_B T)^{2+\sigma}$$
$$= \Gamma\big(\tfrac{10}{3}\big)\, \zeta\big(\tfrac{10}{3}\big)\, A\, (k_B T)^{10/3} \, .$$

Thus,

$$C_V = \Gamma\big(\tfrac{13}{3}\big)\, \zeta\big(\tfrac{10}{3}\big)\, A\, (k_B T)^{7/3} \, ,$$

where we have used $z\,\Gamma(z) = \Gamma(z+1)$.

(c) The pressure is $p = -\Omega/V$, which is

$$p(T,z) = -k_B T \int_{-\infty}^{\infty} d\varepsilon \; g(\varepsilon)\, \ln\big(1 - z\, e^{-\varepsilon/k_B T}\big) = -A\, k_B T \int_0^{\infty} d\varepsilon \; \varepsilon^{\sigma}\, \ln\big(1 - z\, e^{-\varepsilon/k_B T}\big)$$
$$= \frac{A}{1+\sigma} \int_0^{\infty} d\varepsilon \, \frac{\varepsilon^{1+\sigma}}{z^{-1}\, e^{\varepsilon/k_B T} - 1} = \Gamma(1+\sigma)\, A\, (k_B T)^{2+\sigma}\, \mathrm{Li}_{2+\sigma}(z)$$
$$= \Gamma\big(\tfrac{7}{3}\big)\, A\, (k_B T)^{10/3}\, \mathrm{Li}_{10/3}(z) \, .$$

(d) The Fermi energy is obtained from

$$n = \int_0^{\varepsilon_F} d\varepsilon \; g(\varepsilon) = \frac{A\, \varepsilon_F^{1+\sigma}}{1+\sigma} \qquad \Rightarrow \qquad \varepsilon_F(n) = \left(\frac{(1+\sigma)\, n}{A}\right)^{\frac{1}{1+\sigma}} = \left(\frac{7n}{3A}\right)^{3/7} .$$

We obtain the pressure from $p = -\big(\frac{\partial E}{\partial V}\big)_N$. The energy is

$$E = V \int_0^{\varepsilon_F} d\varepsilon \; g(\varepsilon)\, \varepsilon = V \cdot \frac{A\, \varepsilon_F^{2+\sigma}}{2+\sigma} \propto V^{-\frac{1}{1+\sigma}} \, .$$

Thus, $p = \frac{1}{1+\sigma} \cdot \frac{E}{V}$, *i.e.*

$$p(n) = \frac{A\, \varepsilon_F^{2+\sigma}}{(1+\sigma)(2+\sigma)} = \tfrac{3}{10} \big(\tfrac{7}{3}\big)^{3/7} A^{-3/7} n^{10/7} \, .$$

(4) Provide brief but substantial answers to the following:

(a) Consider a three-dimensional gas of N classical particles of mass m in a uniform gravitational field g. Assume $z \geq 0$ and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. Find the heat capacity C_V.
[7 points]

(b) Consider a system with a single phase space coordinate ϕ which lives on a circle. Now consider three dynamical systems on this phase space:

$$\text{(i) } \dot{\phi} = 0 \quad , \quad \text{(ii) } \dot{\phi} = 1 \quad , \quad \text{(iii) } \dot{\phi} = 2 - \cos\phi \ .$$

For each of these systems, tell whether it is recurrent, ergodic, both, or neither, and explain your reasoning.
[6 points]

(c) Explain Boltzmann's H-theorem.
[6 points]

(d) ν moles of gaseous Argon at an initial temperature $T_{\rm A}$ and volume $V_{\rm A} = 1.0\,{\rm L}$ undergo an adiabatic free expansion to an intermediate state of volume $V_{\rm B} = 2.0\,{\rm L}$. After coming to equilibrium, this process is followed by a reversible adiabatic expansion to a final state of volume $V_{\rm C} = 3.0\,{\rm L}$. Let $S_{\rm A}$ denote the initial entropy of the gas. Find the temperatures $T_{\rm B,C}$ and the entropies $S_{\rm B,C}$. Then repeat the calculation assuming the first expansion (from A to B) is a reversible adiabatic expansion and the second (from B to C) an adiabatic free expansion.
[6 points]

Solution :

(a) The partition function is

$$Z = \frac{A^N}{N!}\left(\lambda_T^{-3}\int\limits_0^\infty \! dz\, e^{-mgz/k_{\rm B}T}\right)^N = \frac{1}{N!}\left(\frac{k_{\rm B}T\,A}{mg\lambda_T^3}\right)^N ,$$

where A is the cross-sectional area. Thus,

$$F = -Nk_{\rm B}T\,\ln\!\left(\frac{k_{\rm B}T\,A}{Nmg\lambda_T^3}\right) - Nk_{\rm B}T \ .$$

We then have

$$C_V = -T\frac{\partial^2 F}{\partial T^2} = \tfrac{5}{2}Nk_{\rm B} \ .$$

(b) Recurrence means a system will come arbitrarily close to revisiting any allowed point in phase space. Ergodicity means time averages may be replaced by phase space averages. With these definitions, we see that

$$\begin{aligned} \text{(i) } \dot{\phi} &= 0 \ \text{ : recurrent but not ergodic} \\ \text{(ii) } \dot{\phi} &= 1 \ \text{ : both recurrent and ergodic} \\ \text{(iii) } \dot{\phi} &= 2 - \cos\phi \ \text{ : recurrent but not ergodic} \ . \end{aligned}$$

If by recurrent we mean "in every neighborhood $\mathcal{N}$ of a point ϕ_0 there exists a point which returns to $\mathcal{N}$ after a finite number of iterations of the τ-advance mapping g_τ, then $\dot{\phi} = 0$ surely is recurrent. because all points remain fixed under these dynamics. With $\dot{\phi} = 1$, we have $\phi(t) = t$, which winds around the phase space with uniform angular frequency. This is both recurrent as well as ergodic. For $\dot{\phi} = 2 - \cos\phi$, we have $\dot{\phi} > 0$ so the motion is constantly winding around the phase space, *i.e.* it doesn't get stuck at a fixed point. So it is recurrent, but not ergodic, because the phase space velocity is relatively slow in the vicinity of $\phi = 0$ and relatively fast in the vicinity of $\phi = \pi$, and time averages will weigh more heavily the neighborhood of $\phi = 0$.

(c) If a probability distribution P_i evolves according to a master equation,

$$\dot{P}_i = \sum_j \left(W_{ji} P_j - W_{ij} P_i \right) ,$$

then one can construct a quantity $\mathsf{H}(t)$ which is a function of the distribution and which satisfies $\dot{\mathsf{H}} \leq 0$. Explicitly, one has

$$\mathsf{H}(t) = \sum_i P_i(t) \ln\Big(P_i(t) / P_i^{\text{eq}} \Big) ,$$

where P_i^{eq} is the equilibrium distribution, which is a fixed point of the master equation. Any such probability distribution therefore evolves *irreversibly*.

(d) Argon is a monatomic ideal gas, thus $\gamma = c_p/c_V = \frac{5}{3}$. The adiabatic equation of state is $d(TV^{\gamma-1}) = 0$. The entropy of a monatomic ideal gas is $S = \frac{3}{2} N k_{\text{B}} \ln(E/N) + N k_{\text{B}} \ln(V/N) + Na$ where a is a constant. During an adiabatic free expansion, $\Delta E = Q = W = 0$. We can now construct the following table:

	T_{B}	T_{C}	$S_{\text{B}} - S_{\text{A}}$	$S_{\text{C}} - S_{\text{A}}$
AB free / BC reversible	T_{A}	$(3/2)^{-2/3}\, T_{\text{A}}$	$\nu R \ln 2$	$\nu R \ln 2$
AB reversible / BC free	$2^{-2/3}\, T_{\text{A}}$	$2^{-2/3}\, T_{\text{A}}$	0	$\nu R \ln(3/2)$

(5) Match the Jonathan Coulton song lines in the left column with their following lines in the right column.
`[30 quatloos extra credit]`

(a) That was a joke – haha – fat chance	(1) I can see the day unfold in front of me
(b) Saw a vision in his head	(2) I'm glad to see you take constructive criticism well
(c) I try to medicate my concentration haze	(3) And this mountain is covered with wolves
(d) I've been patient, I've been gracious	(4) A bulbous pointy form
(e) I guess we'll table this for now	(5) Hearing the whirr of the servos inside
(f) She'll eye me suspiciously	(6) Anyway this cake is great

Solution :
(a) 6   (b) 4   (c) 1   (d) 3   (e) 2   (f) 5