PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #3

(1) Consider a generalization of the situation in §4.4 of the notes where now three reservoirs are in thermal contact, with any pair of systems able to exchange energy.

(a) Assuming interface energies are negligible, what is the total density of states $D(E)$? Your answer should be expressed in terms of the densities of states functions $D_{1,2,3}$ for the three individual systems.

(b) Find an expression for $P(E_1, E_2)$, which is the joint probability distribution for system 1 to have energy E_1 while system 2 has energy E_2 and the total energy of all three systems is $E_1 + E_2 + E_3 = E$.

(c) Extremize $P(E_1, E_2)$ with respect to $E_{1,2}$. Show that this requires the temperatures for all three systems must be equal: $T_1 = T_2 = T_3$. Writing $E_j = E_j^* + \delta E_j$, where E_j^* is the extremal solution ($j = 1, 2$), expand $\ln P(E_1^* + \delta E_1 \, , \, E_2^* + \delta E_2)$ to second order in the variations δE_j. Remember that

$$S = k_{\rm B} \ln D \quad , \quad \left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T} \quad , \quad \left(\frac{\partial^2 S}{\partial E^2}\right)_{V,N} = -\frac{1}{T^2 C_V} \ .$$

(d) Assuming a Gaussian form for $P(E_1, E_2)$ as derived in part (c), find the variance of the energy of system 1,

$$\mathsf{Var}(E_1) = \big\langle (E_1 - E_1^*)^2 \big\rangle \ .$$

(2) Consider a two-dimensional gas of identical classical, noninteracting, massive relativistic particles with dispersion $\varepsilon(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 c^2 + m^2 c^4}$.

(a) Compute the free energy $F(T, V, N)$.

(b) Find the entropy $S(T, V, N)$.

(c) Find an equation of state relating the fugacity $z = e^{\mu/k_{\rm B}T}$ to the temperature T and the pressure p.

(3) A three-level system has energy levels $\varepsilon_0 = 0$, $\varepsilon_1 = \Delta$, and $\varepsilon_2 = 4\Delta$. Find the free energy $F(T)$, the entropy $S(T)$ and the heat capacity $C(T)$.

(4) Consider a many-body system with Hamiltonian $\hat{H} = \frac{1}{2} \hat{N} (\hat{N} - 1)\, U$, where $\hat{N}$ is the particle number and $U > 0$ is an interaction energy. Assume the particles are identical and can be described using Maxwell-Boltzmann statistics, as we have discussed. Assuming $\mu = 0$, plot the entropy S and the average particle number N as functions of the scaled temperature $k_{\rm B}T/U$. (You will need to think about how to impose a numerical cutoff in your calculations.)