PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #8 SOLUTIONS

(1) For the Dieterici equation of state,

$$p(V - Nb) = Nk_{\rm B}T\, e^{-Na/Vk_{\rm B}T} ,$$

find the virial coefficients $B_2(T)$ and $B_3(T)$.

Solution :

We first write the equation of state as $p = (n, T)$ where $n = N/V$:

$$p = \frac{nk_{\rm B}T}{1 - bn}\, e^{-an/k_{\rm B}T} .$$

Next, we expand in powers of the density n:

$$\begin{aligned} p &= nk_{\rm B}T\,\big(1 + bn + b^2n^2 + \ldots\big)\,\big(1 - \beta an + \tfrac{1}{2}\beta^2a^2n^2 + \ldots\big) \\ &= nk_{\rm B}T\Big[1 + \big(b - \beta a\big)\,n + \big(b^2 - \beta ab + \tfrac{1}{2}\beta^2a^2\big)\,n^2 + \ldots\Big] \\ &= nk_{\rm B}T\Big[1 + B_2\,n + B_3\,n^2 + \ldots\Big] , \end{aligned}$$

where $\beta = 1/k_{\rm B}T$. We can now read off the virial coefficients:

$$B_2(T) = b - \frac{a}{k_{\rm B}T} \qquad , \qquad B_3 = b^2 - \frac{ab}{k_{\rm B}T} + \frac{a^2}{2k_{\rm B}^2T^2} .$$

(2) Consider a gas of particles with dispersion $\varepsilon(\boldsymbol{k}) = \varepsilon_0\,|\boldsymbol{k}\ell|^{5/2}$, where ε_0 is an energy scale and ℓ is a length scale.

(a) Find the density of states $g(\varepsilon)$ in $d = 2$ and $d = 3$ dimensions.

(b) Find the virial coefficients $B_2(T)$ and $B_3(T)$ in $d = 2$ and $d = 3$ dimensions.

(c) Find the heat capacity $C_V(T)$ in $d = 3$ dimensions for photon statistics.

Solution :

(a) For $\varepsilon(\boldsymbol{k}) = \varepsilon_0\,|\boldsymbol{k}\ell|^{\alpha}$ we have

$$\begin{aligned} g(\varepsilon) = \int\!\frac{d^dk}{(2\pi)^d}\,\delta\big(\varepsilon - \varepsilon(\boldsymbol{k})\big) &= \frac{\Omega_d}{(2\pi)}\int\limits_0^\infty\! dk\; k^{d-1}\,\frac{\delta\big(k - (\varepsilon/\varepsilon_0)^{1/\alpha}/\ell\big)}{\alpha\varepsilon_0\ell^\alpha\,k^{\alpha-1}} \\ &= \frac{\Omega_d}{(2\pi)^d}\,\frac{1}{\alpha\varepsilon_0\ell^d}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{d}{\alpha}-1}\Theta(\varepsilon) . \end{aligned}$$

Thus, for $\alpha = \frac{5}{2}$,

$$g_{d=2}(\varepsilon) = \frac{1}{5\pi\varepsilon_0\ell^2} \left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1/5} \Theta(\varepsilon) \qquad , \qquad g_{d=3}(\varepsilon) = \frac{1}{5\pi\varepsilon_0\ell^3} \left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/5} \Theta(\varepsilon) \,.$$

(b) We must compute the coefficients

$$\begin{aligned} C_j &= \int\limits_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) \, e^{-j\varepsilon/k_B T} = \frac{\Omega_d}{(2\pi)^d} \frac{1}{\alpha\varepsilon_0\ell^d} \int\limits_{0}^{\infty} d\varepsilon \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{d}{\alpha}-1} e^{-j\varepsilon/k_B T} \\ &= \frac{\Omega_d \, \Gamma(d/\alpha)}{(2\pi)^d} \frac{1}{\alpha\ell^d} \left(\frac{k_B T}{j\varepsilon_0}\right)^{d/\alpha} \equiv j^{-d/\alpha} \, \lambda_T^{-d} \,, \end{aligned}$$

where

$$\lambda_T \equiv \frac{2\pi\ell}{\left[\Omega_d \, \Gamma\big(\frac{d}{\alpha}\big)/\alpha\right]^{1/d}} \left(\frac{\varepsilon_0}{k_B T}\right)^{1/\alpha} .$$

Then

$$\begin{aligned} B_2(T) &= \mp \frac{C_2}{2\,C_1^2} = \mp 2^{-\left(\frac{d}{\alpha}+1\right)} \, \lambda_T^d \\ B_3(T) &= \frac{C_2^2}{C_1^4} - \frac{2\,C_3}{C_1^3} = \left[4^{-\frac{d}{\alpha}} - \tfrac{2}{3} \cdot 3^{-\frac{d}{\alpha}}\right] \lambda_T^{2d} \,. \end{aligned}$$

We have $\alpha = \frac{5}{2}$, so $\frac{d}{\alpha} = \frac{4}{5}$ for $d = 2$ and $\frac{6}{5}$ for $d = 3$.

(c) For photon statistics, the energy is

$$E(T,V) = V \int\limits_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) \, \varepsilon \, \frac{1}{e^{\varepsilon/k_B T} - 1} = \frac{V \, \Omega_d \, \varepsilon_0}{(2\pi\ell)^d \, \alpha} \, \Gamma\big(\tfrac{d}{\alpha}+1\big) \zeta\big(\tfrac{d}{\alpha}+1\big) \left(\frac{k_B T}{\varepsilon_0}\right)^{\frac{d}{\alpha}+1}$$

Thus,

$$C_V = \frac{\partial E}{\partial T} = \frac{V \, \Omega_d \, k_B}{(2\pi\ell)^d \, \alpha} \, \Gamma\big(\tfrac{d}{\alpha}+2\big) \zeta\big(\tfrac{d}{\alpha}+1\big) \left(\frac{k_B T}{\varepsilon_0}\right)^{\frac{d}{\alpha}} .$$

(3) At atmospheric pressure, what would the temperature T have to be in order that the electromagnetic energy density should be identical to the energy density of a monatomic ideal gas?

Solution :

The pressure is $p = 1.0\,\text{atm} \simeq 10^5\,\text{Pa}$. We set

$$\frac{E}{V} = \tfrac{3}{2}\, p = \frac{2\pi^2}{30} \, \frac{(k_B T)^4}{(\hbar c)^3} \,,$$

and solve for T:

$$T = \frac{1}{1.38\times 10^{-23}\,\mathrm{J/K}} \cdot \left[\frac{45}{2\pi^2}\cdot (10^5\,\mathrm{Pa})\cdot \Big(1970\,\mathrm{eV\,Å}\cdot 1.602\times 10^{-19}\,\frac{\mathrm{J}}{\mathrm{eV}}\cdot 10^{-10}\,\frac{\mathrm{m}}{\mathrm{Å}}\Big)^3\right]^{1/4}$$
$$= 1.19\times 10^5\,\mathrm{K}\ .$$

(4) Find the internal energy and heat capacity for a two-dimensional crystalline insulator, according to the Debye model.

Solution :

We have

$$\Omega(T,V) = Nk_{\mathrm{B}}T\int\limits_0^\infty \!d\omega\; g(\omega)\;\ln\left[2\sinh\left(\frac{\hbar\omega}{2k_{\mathrm{B}}T}\right)\right]\ .$$

The internal energy is given by

$$E(T,V) = \frac{\partial(\beta\Omega)}{\partial\beta} = \tfrac{1}{2}\,N\int\limits_0^\infty \!d\omega\; g(\omega)\,\hbar\omega\;\mathrm{ctnh}\left(\frac{\hbar\omega}{2k_{\mathrm{B}}T}\right)\ .$$

In the three-dimensional Debye model, the phonon density of states per unit cell is

$$g(\omega) = \frac{9\omega^2}{\omega_{\mathrm{D}}^3}\,\Theta(\omega_{\mathrm{D}}-\omega)\ ,$$

where ω_{D} is the Debye frequency. Thus,

$$E(T) = \frac{9N\hbar}{2\omega_{\mathrm{D}}^3}\int\limits_0^{\omega_{\mathrm{D}}}\!d\omega\;\omega^3\;\mathrm{ctnh}\left(\frac{\hbar\omega}{2k_{\mathrm{B}}T}\right)$$
$$= \frac{72N}{(\hbar\omega_{\mathrm{D}})^3}\,(k_{\mathrm{B}}T)^4\int\limits_0^{\frac{\hbar\omega_{\mathrm{D}}}{2k_{\mathrm{B}}T}}\!ds\;s^3\;\mathrm{ctnh}\,(s)\ .$$

In $d=2$ dimensions, we must replace the phonon density of states with

$$g(\omega) = \frac{4\omega}{\omega_{\mathrm{D}}^2}\,\Theta(\omega_{\mathrm{D}}-\omega)\ .$$

This guarantees that the integrated phonon density of states per unit cell is 2, which is the number of acoustic phonon modes in two dimensions. We then have

$$E(T) = \frac{2\hbar}{\omega_{\rm D}^2}\, N \int\limits_0^{\omega_{\rm D}} d\omega\, \omega^2\, \operatorname{ctnh}\!\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)$$

$$= \frac{16N}{(\hbar\omega_{\rm D})^2}\, (k_{\rm B}T)^3 \int\limits_0^{\frac{\hbar\omega_{\rm D}}{2k_{\rm B}T}} ds\, s^2\, \operatorname{ctnh}(s)\ .$$

The heat capacity is

$$C_V = \frac{\partial E}{\partial T} = \frac{N\hbar^2}{k_{\rm B}T^2\omega_{\rm D}^2} \int\limits_0^{\omega_{\rm D}} d\omega\, \omega^3\, \operatorname{csch}^2\!\left(\frac{\hbar\omega}{2k_{\rm B}T}\right)$$

$$= 16Nk_{\rm B}\left(\frac{k_{\rm B}T}{\hbar\omega_{\rm D}}\right)^2 \int\limits_0^{\frac{\hbar\omega_{\rm D}}{2k_{\rm B}T}} ds\, s^2\, \operatorname{csch}^2(s)\ .$$

One can check that $\lim_{T\to\infty} C_V(T) = 2Nk_{\rm B}$, which is the appropriate Dulong-Petit limit.