PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #9 SOLUTIONS

(1) For a system of noninteracting $S=0$ bosons obeying the dispersion $\varepsilon(\boldsymbol{k}) = \hbar v|\boldsymbol{k}|$.

(a) Find the density of states per unit volume $g(\varepsilon)$.

(b) Determine the critical temperature for Bose-Einstein condensation in three dimensions.

(c) Find the condensate fraction n_0/n for $T < T_c$.

(d) For this dispersion, is there a finite transition temperature in $d=2$ dimensions? If not, explain why. If so, compute $T_c^{(d=2)}$.

Solution :

(a) The density of states in d dimensions is

$$g(\varepsilon) = \int \frac{d^d k}{(2\pi)^d}\,\delta(\varepsilon - \hbar v k) = \frac{\Omega_d}{(2\pi)^d}\,\frac{\varepsilon^{d-1}}{(\hbar v)^d}\ .$$

(b) The condition for $T = T_c$ is to write $n = n(T_c, \mu = 0)$:

$$n = \int_0^\infty \!d\varepsilon\,\frac{g(\varepsilon)}{e^{\varepsilon/k_B T_c} - 1} = \frac{1}{2\pi^2(\hbar v)^3}\int_0^\infty \!d\varepsilon\,\frac{\varepsilon^2}{e^{\varepsilon/k_B T_c} - 1} = \frac{\zeta(3)}{\pi^2}\left(\frac{k_B T_c}{\hbar v}\right)^3 .$$

Thus,

$$k_B T_c = \left(\frac{\pi^2}{\zeta(3)}\right)^{1/3} \hbar v\, n^{1/3}\ .$$

(c) For $T < T_c$, we have

$$n = n_0 + \frac{\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar v}\right)^3 .$$

Thus,

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_c(n)}\right)^3 .$$

(d) In $d=2$ we have

$$n = \frac{1}{2\pi(\hbar v)^2}\int_0^\infty \!d\varepsilon\,\frac{\varepsilon}{e^{\varepsilon/k_B T_c} - 1} = \frac{\zeta(2)}{2\pi}\left(\frac{k_B T_c}{\hbar v}\right)^2$$

and hence

$$k_{\mathrm{B}} T_{\mathrm{c}}^{(d=2)} = \hbar v \sqrt{\frac{2\pi n}{\zeta(2)}} \, .$$

(2) Using the argument we used in class and in §5.4.2 of the notes, predict the surface temperatures of the remaining planets in our solar system. In each case, compare your answers with the most reliable source you can find. In cases where there are discrepancies, try to come up with a convincing excuse.

Solution :

Relavant planetary data are available from

`http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html`

and from Wikipedia. According to the derivation in the notes, we have

$$T = \left(\frac{R_{\odot}}{2a}\right)^{1/2} T_{\odot} \, ,$$

where $R_{\odot} = 6.96 \times 10^5$ km and $T_{\odot} = 5780$ K. From this equation and the reported values for a for each planet, we obtain the following table:

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
a (10^8 km)	0.576	1.08	1.50	2.28	7.78	14.3	28.7	45.0	59.1
$T_{\mathrm{surf}}^{\mathrm{obs}}$ (K)	340^*	$735^{\dagger}$	$288^{\ddagger}$	210	112	84	53	55	44
$T_{\mathrm{surf}}^{\mathrm{pred}}$ (K)	448	327	278	226	122	89.1	63.6	50.8	44.3

Table 1: Planetary data from GSU web site and from Wikipedia. Observed temperatures are averages. $*$ mean equatorial temperature. † mean temperature below cloud cover.

Note that we have included Pluto, because since my childhood Pluto has always been the ninth planet to me. We see that our simple formula works out quite well except for Mercury and Venus. Mercury, being so close to the sun, has enormous temperature fluctuations as a function of location. Venus has a whopping greenhouse effect.

(3) Read carefully the new and improved §5.5.4 of the lecture notes ("Melting and the Lindemann criterion"). Using the data in Table 5.1, and looking up the atomic mass and lattice constant of tantalum (Ta), find the temperature T_{L} where the Lindemann criterion predicts Ta should melt.

Solution :

One finds the mass of tantalum is $M = 181\,\mathtt{amu}$, and the lattice constant is $a = 3.30\,\text{Å}$. Thus,

$$\Theta^\star = \frac{109\,\text{K}}{M[\mathtt{amu}]\big(a[\text{Å}]\big)^2} = 55.3\,\text{mK} \ .$$

From the table in the lecture notes, the Debye temperature is $\Theta_{\text{D}} = 246\,\text{K}$ and the melting point is $T_{\text{melt}} = 2996\,\text{K}$. The Lindemann temperature is

$$T_{\text{L}} = \left(\frac{\eta^2\,\Theta_{\text{D}}}{\Theta^\star} - 1\right)\frac{\Theta_{\text{D}}}{4} = 2674\,\text{K} \ ,$$

where $\eta = 0.10$. Close enough for government work.

(4) For ideal Fermi gases in $d = 1$, 2, and 3 dimensions, compute at $T = 0$ the average fermion velocity.

Solution :

At $T = 0$ the average velocity is

$$\langle v \rangle = \int\limits_0^{k_{\text{F}}}\! dk\; k^{d-1}\,\frac{\hbar k}{m} \bigg/ \int\limits_0^{k_{\text{F}}}\! dk\; k^{d-1} = \frac{d}{d+1}\cdot\frac{\hbar k_{\text{F}}}{m} \ .$$

The number density is

$$n = \frac{\mathsf{g}\,\Omega_d}{(2\pi)^d}\int\limits_0^{k_{\text{F}}}\! dk\; k^{d-1} = \frac{\mathsf{g}\,\Omega_d\,k_{\text{F}}^d}{(2\pi)^d\,d} \qquad \Rightarrow \qquad k_{\text{F}} = 2\pi\left(\frac{d}{\mathsf{g}\,\Omega_d}\right)^{1/d} n^{1/d} \ .$$

Putting these together we can obtain the average velocity in terms of the density n and physical constants. (OK! OK! I mean average speed!)

(5) Consider a three-dimensional Fermi gas of $S = \frac{1}{2}$ particles obeying the dispersion relation $\varepsilon(\mathbf{k}) = A\,|\mathbf{k}|^4$.

(a) Compute the density of states $g(\varepsilon)$.

(b) Compute the molar heat capacity.

(c) Compute the lowest order nontrivial temperature dependence for $\mu(T)$ at low temperatures. *I.e.* compute the $\mathcal{O}(T^2)$ term in $\mu(T)$.

Solution :

(a) The density of statesin $d = 3$ ($\mathrm{g} = 2S + 1 = 2$) is given by

$$g(\varepsilon) = \frac{1}{\pi^2} \int_0^\infty dk\, k^2\, \delta\big(\varepsilon - \varepsilon(k)\big) = \frac{1}{\pi^2}\, k^2(\varepsilon)\, \frac{dk}{d\varepsilon}\bigg|_{k=(\varepsilon/A)^{1/4}} = \frac{\varepsilon^{-1/4}}{4\pi^2 A^{3/4}}\,.$$

(b) The molar heat capacity is

$$c_V = \frac{\pi^2}{3n}\, R\, g(\varepsilon_{\mathrm{F}})\, k_{\mathrm{B}} T = \frac{\pi^2 R}{4} \cdot \frac{k_{\mathrm{B}} T}{\varepsilon_{\mathrm{F}}}\,,$$

where $\varepsilon_{\mathrm{F}} = \hbar^2 k_{\mathrm{F}}^2/2m$ can be expressed in terms of the density using $k_{\mathrm{F}} = (3\pi^2 n)^{1/3}$, which is valid for any isotropic dispersion in $d = 3$. In deriving this formula we had to express the density n, which enters in the denominator in the above expression, in terms of ε_{F}. But this is easy:

$$n = \int_0^{\varepsilon_{\mathrm{F}}} d\varepsilon\; g(\varepsilon) = \frac{1}{3\pi^2} \left(\frac{\varepsilon_{\mathrm{F}}}{A}\right)^{3/4}.$$

(c) We have (Lecture Notes, §5.7.5)

$$\delta\mu = -\frac{\pi^2}{6}\, (k_{\mathrm{B}} T)^2\, \frac{g'(\varepsilon_{\mathrm{F}})}{g(\varepsilon_{\mathrm{F}})} = \frac{\pi^2}{24} \cdot \frac{(k_{\mathrm{B}} T)^2}{\varepsilon_{\mathrm{F}}}\,.$$

Thus,

$$\mu(n,T) = \varepsilon_{\mathrm{F}}(n) + \frac{\pi^2}{24} \cdot \frac{(k_{\mathrm{B}} T)^2}{\varepsilon_{\mathrm{F}}(n)} + \mathcal{O}(T^4)\,,$$

where $\varepsilon_{\mathrm{F}}(n) = \frac{\hbar^2}{2m}\, (3\pi^2 n)^{2/3}$.