Problem 1

(a) The total charge on the inner surface is $-Q$, so that $E=0$ inside metal

$\Rightarrow$ $\boxed{\sigma_i = \dfrac{-Q}{4\pi R^2}}$; on outer surface, $+Q$ since metal shell is neutral

$\Rightarrow$ $\boxed{\sigma_o = \dfrac{+Q}{16\pi R^2}}$

(b) By Gauss' law, electric field at P is not affected by outer charges

$\Rightarrow$ $E_p = 3\ N/C$ with metal shell also.

(c) Both with and without metal shell, field at P' is the same

$$E_{p'} = \frac{Q}{4\pi\varepsilon_0 (3R)^2} = E_p \cdot \frac{R^2}{(3R)^2} = \frac{E_p}{9} \Rightarrow \boxed{E_{p'} = \frac{1}{3}\ N/C}$$

(d) If Q is not at center of the shell:

(i) electric field at P does change.

(ii) inner surface charge density changes. It is no longer uniform. Larger close to Q.

(iii) outer surface charge density does not change, because $E=0$ inside shell, so outer charge doesn't know where inner charge is.

(iv) $E_{p'}$ doesn't change, for same reason as (iii).

Problem 2

$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$. Take cylindrical surface of height h.

(a) (i) For $r < R$, $E = 0$

(ii) $R < r < 2R$: $E \cdot 2\pi r \cdot h = \frac{\rho}{\varepsilon_0} h \cdot \pi (r^2 - R^2) \Rightarrow$

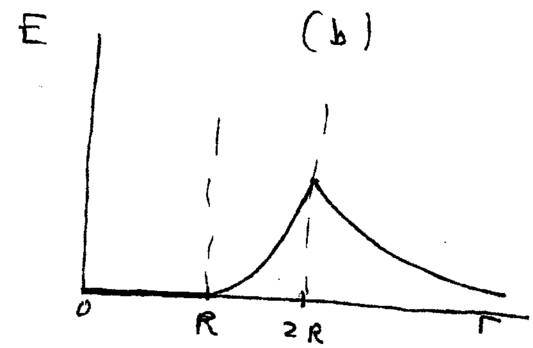
$$\boxed{E = \frac{\rho}{2\varepsilon_0} \frac{r^2 - R^2}{r}}$$

(iii) $2R < r$:

$E \cdot 2\pi r \cdot h = \frac{\rho}{\varepsilon_0} \cdot h \cdot \pi ((2R)^2 - R^2) \Rightarrow$

$$\boxed{E = \frac{3\rho R^2}{2\varepsilon_0 r}}$$

(b)



(c) For $0 < r < R$, $V = 0$ (no electric field)

For $R < r < 2R$: $V(r) = -\int_R^r E(r')dr' = -\frac{\rho}{2\varepsilon_0} \int_R^r dr' \frac{r'^2 - R^2}{r'} =$

$$= -\frac{\rho}{2\varepsilon_0} \int_R^r dr' \left(r' - \frac{R^2}{r'}\right) = -\frac{\rho}{2\varepsilon_0} \left(\frac{r'^2}{2} - R^2 \ln r'\right)\Big|_R^r = -\frac{\rho}{2\varepsilon_0}\left(\frac{r^2 - R^2}{2} - R^2 \ln \frac{r}{R}\right)$$

So $\boxed{V(r) = -\frac{\rho}{2\varepsilon_0}\left[\frac{r^2 - R^2}{2} - R^2 \ln \frac{r}{R}\right] \quad \text{always} < 0}$

(d) Work required to bring charge q from r_2 to r_1, both $> 2R$:

$$W = q\int_{r_1}^{r_2} E(r)\,dr = q \int_{r_1}^{r_2} \frac{3\rho R^2}{2\varepsilon_0} \frac{dr}{r} = q \cdot \frac{3\rho R^2}{2\varepsilon_0} \ln \frac{r_2}{r_1} \equiv C \ln \frac{r_2}{r_1}$$

For $r_1 = 3R$, $r_2 = 4R$: $W = C \ln \frac{4}{3} = 5J$

For $r_1 = 2R$, $r_2 = 3R$: $W' = C \ln \frac{3}{2} = 5J \cdot \frac{\ln \frac{3}{2}}{\ln \frac{4}{3}} = \boxed{7.05\ J}$

Problem 3

Discharge in RC circuit:

$$Q(t) = Q_0 e^{-t/RC} \quad , \quad I(t) = \frac{Q_0}{RC} e^{-t/RC}$$

Here the time constant is $\tau_c = RC = 100\,\Omega \times 1F = 100s$.

$Q_0 = 1C$ (a) $\boxed{I(t=0^+) = \frac{1C}{100s} = 10mA}$

(b)

Energy dissipated during time t: 2 ways to calculate it

(i) After 1 second, $Q(t=1s) = Q_0 e^{-1s/RC}$. So capacitor lost energy, that was dissipated:

$$\Delta U = \frac{Q_0^2}{2C} - \frac{Q(t=1s)^2}{2C} = \frac{Q_0^2}{2C}\left(1-e^{-2t/RC}\right) = \frac{1}{2 \cdot 1}\left(1-e^{-2/100}\right) = 0.099\,J$$

$\boxed{\text{So } 0.0099\,J \text{ was dissipated}} \boxed{= 9.9mJ}$

(ii) Same result is obtained from $\Delta U = \int_0^t dt\, I(t)^2 \cdot R$, verify it.

Compare with $I(t=0^+)^2 \cdot R \times 1s = (0.01A)^2 \cdot 100\,\Omega \times 1s = \boxed{0.01J = 10mJ}$

Answers are similar because the time interval $t = 1s$ is much smaller than the time constant $RC = 100s$, so I stays approximately constant.

(c) After a long time passed, energy dissipated is total energy in capacitor:

$$\boxed{U = \frac{Q_0^2}{2C} = 0.5J}$$

(d) If we put in dielectric, $\boxed{U = \frac{Q_0^2}{2KC} = \frac{0.5J}{5} = 0.1J}$ is the energy dissipated in the resistor

(e) When the dielectric is put in, the capacitor does work on the agent putting in the dielectric, and loses energy. Or, the dielectric will speed up and lose kinetic energy by friction, dissipating heat.

Problem 4

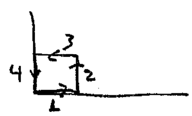
Calculate $\vec{\nabla} \times \vec{E}$. If it is non-zero, it is not an electrostatic field.

$$\vec{\nabla} \times \vec{E} = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{y_0} & -\frac{x}{x_0} & 0 \end{pmatrix} = \hat{k}\left(-\frac{1}{x_0} - \frac{1}{y_0}\right)\frac{V}{m} \neq 0$$

(b) Faraday law: $\frac{\partial B}{\partial t} = -\vec{\nabla} \times \vec{E} = \left(\frac{1}{x_0} + \frac{1}{y_0}\right)\hat{k}\,\frac{V}{m} = 2\,\frac{T}{s}\,\hat{k}$

$x_0 = y_0 = 1m \Rightarrow B(t) = \int_0^t \frac{\partial B}{\partial t'}\,dt' = \frac{2T}{s}.t\Big]_0^{1s}$

$\Rightarrow \boxed{B(t=1s) = 2T}$ in the $+\hat{k}$ direction

(c) 

The emf is: $\mathcal{E} = \oint \vec{E}.d\vec{l} = -\int \frac{\partial B}{\partial t}\,dA = -\frac{\partial B}{\partial t}.A$

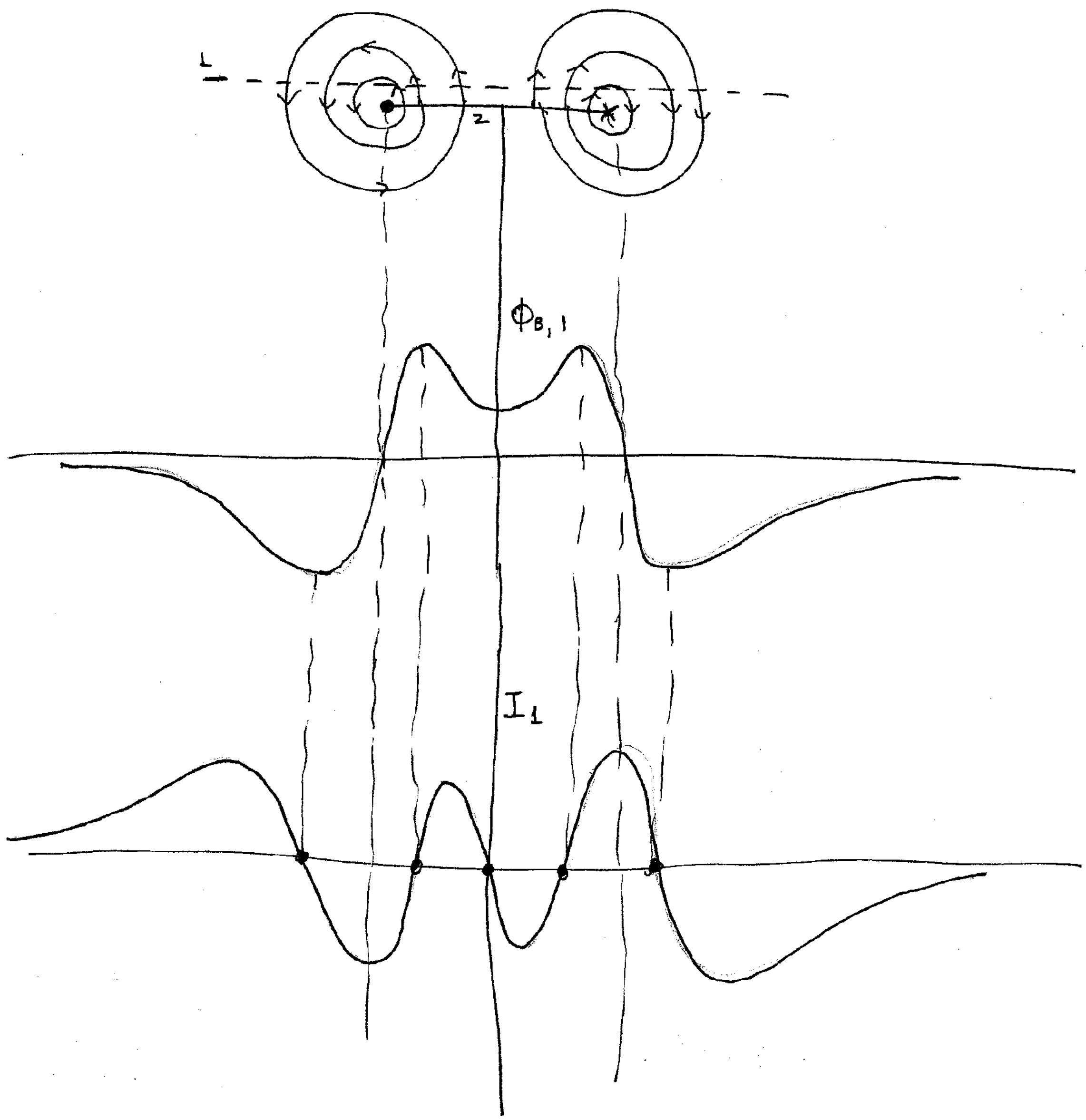
Since $A = 1m^2$, $\boxed{\mathcal{E} = 2V.}$

Or, we can do $\mathcal{E} = \int_2 E_y\,dl + \int_3 E_x.dl = \int_0^1 -\frac{x}{x_0}\,dy + \int_1^0 \frac{y}{y_0}\,dx = -2V$

emf generates current flow clockwise seen from a point at $z > 0$.

Current is $I = \frac{\mathcal{E}}{R} = \frac{2V}{2\Omega} \Rightarrow \boxed{I = 1A}$

Problem 5

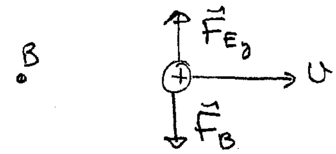


For x large and negative: flux is negative and becoming more negative $\Rightarrow$ by Lenz's law, current will create positive flux in loop 1 $\Rightarrow$ flows in same direction as current in loop 2 $\Rightarrow$ positive by convention.

Where Φ_{B1} has maximum or minimum, it is not changing $\Longrightarrow$ no current induced.

Note: Φ_{B1} is even, I_1 is odd function of x

Problem 6



$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B}$$

$$F_B = q\,\vec{v}\times\vec{B}$$

(a) Since the proton continues to move in the x direction and the B-field does not exert any force in the z direction $\Rightarrow$ $\boxed{E_z = 0}$

(b) Since there is no deflection in y direction, electric force exactly cancels magnetic force $\Rightarrow$ $q E_y = q\, v_x B$ $\Rightarrow$ at $x = 0$, $v_x = 10{,}000\ \frac{m}{s}$

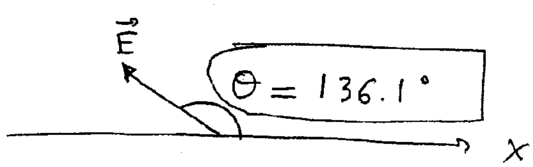
$$E_y = vB = 10{,}000 \cdot 0.5\times 10^{-4}\ \frac{V}{m} \Rightarrow \boxed{E_y = 0.5\ \frac{V}{m}}$$

(c)

The proton stops after 1m. Equate potential and kinetic energies: $\frac{1}{2} m v^2 = q\, E_x\, d \Rightarrow$

$$E_x = \frac{1}{2}\,\frac{m v^2}{q d} = \frac{1}{2}\cdot\frac{1.67\times 10^{-27}\cdot 10{,}000^2}{1.6\times 10^{-19}\cdot 1}\ V/m$$

$\Rightarrow$ $E_x = 0.52$ V/m in the negative x direction



(at $x = 0$)

$$\tan\theta = -\frac{0.5}{0.52}$$

Problem 7

The resistance in the circuit is

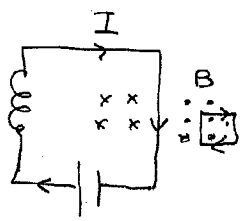
$$R = \frac{\rho \cdot l}{A} \quad , \; l = 10\,m, \; A = 10^{-6}\,m^2, \; \rho = 1.68\times 10^{-8}\,\Omega m$$

$$\Rightarrow \boxed{R = 0.168\,\Omega} \quad 2$$

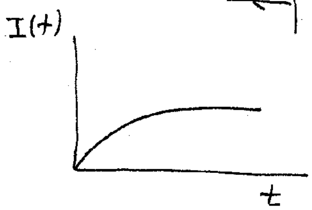
The time constant for the R-L circuit is

$$\tau_L = L/R \; , \; L = 100\,mH = 0.1\,H \;\Rightarrow\; \boxed{\tau_L = 0.595\,s} \quad 1$$

So we have



B increases with t $\Rightarrow$ current induced in small loop generates B into the paper



$\Rightarrow$ Current in small loop flows clockwise 3

$$I(t) = \frac{V_0}{R}\left(1 - e^{-t/\tau_L}\right)$$

At t = 1s, current in small loop is 1 μA. That comes from induced emf

$$\mathcal{E} = -\frac{d}{dt}\int B \cdot dA \quad , \quad \text{now } B \propto I \Rightarrow \frac{dB}{dt} \propto \frac{V_0}{R\tau_L} e^{-t/\tau_L}$$

So we have $I(\text{small loop, time } t) = C\, e^{-t/\tau_L}$

For $t = 1s$, $I = 1\,\mu A \Rightarrow C = 1\,\mu A / e^{-1/0.595} = 5.37\,\mu A$

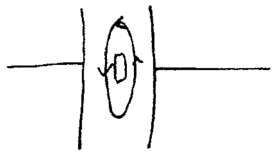
Hence:

$$\boxed{\begin{aligned} I(t=2s) &= 5.37\, e^{-2/0.595}\,\mu A = 0.186\,\mu A \\ I(t=0s) &= 5.37\, e^{0}\,\mu A = 5.37\,\mu A \end{aligned}} \quad 4$$

Problem 8

$$Q(t) = Q_0 \left(\frac{t}{\tau}\right)^2$$

There is an induced magnetic field according to Maxwell law.



If the flux of the magnetic fields through the loops changes with time, there will be an emf induced in the loops and a current, by Faraday law.

Loop 1 is on the axis $\Rightarrow$ there is no net flux of B-field through it $\Rightarrow$ no current in loop 1. Yes in loop 2.

Displacement current is $I_d = \dfrac{dQ}{dt} = \dfrac{2Q_0 t}{\tau^2} = \dfrac{2 \cdot 1 \cdot t}{10^{-6}} = 2 \cdot 10^6\ A \cdot t(s)$

The magnetic field at the position of loop 2 is:

$$B = \frac{\mu_0 I_d}{2\pi a} = \frac{4\pi \times 10^{-7} \cdot 2 \times 10^6}{2\pi \times 0.1} \cdot t \cdot T = 4\,T \cdot t(s)$$

The flux of B through loop 2 is $\Phi_B = 4 \cdot 0.01^2 \cdot t(s)\ T.m^2 =$

$$= 0.0004\, t(s)\ T.m^2$$

So the induced emf is $\mathcal{E} = \dfrac{\partial \Phi_B}{\partial t} = 0.0004\ V$

Since $R_2 = 10\,\Omega \Rightarrow I_2 = \dfrac{0.0004V}{10\,\Omega} \Rightarrow \boxed{I_2 = 40\mu A}$

(c) If the exponent $m = 1$, the B resulting from Maxwell's law is time independent $\Rightarrow$ magnetic flux through loop is constant $\Rightarrow$ no induced emf by Faraday $\Rightarrow$ no current.