

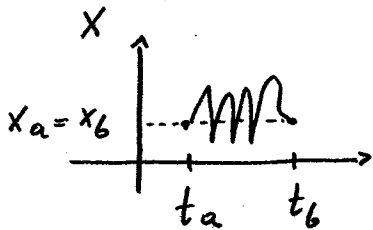
# Path Integral and Statistical Physics

Consider the trace relation

$$\text{Tr } K = \sum_n e^{-\frac{i}{\hbar} E_n t}$$

$$t = t_b - t_a$$

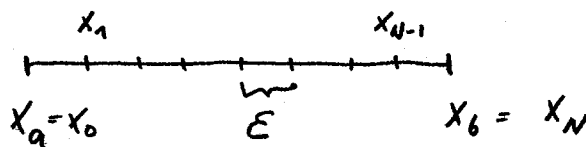
$$\hookrightarrow \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar} S}$$



sum over all closed paths  
(periodic paths with period  $t$ )

$$K(b, a) = \lim_{\epsilon \rightarrow 0} \int dx_1 \dots dx_{N-1} \left( \frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{N}{2}} x$$

$$\exp \left\{ \frac{im}{2\hbar\epsilon} \sum_{i=1}^N (x_i - x_{i-1})^2 - \frac{i}{\hbar} \epsilon \sum_{i=1}^N V \left( \frac{x_{i-1} + x_i}{2} \right) \right\}$$



$x_a$  and  $x_b$  are different and not integrated in  $K(b, a)$

$$\text{Tr } K = \int dx_a K(a, a)$$

one extra integration

$$x_a = x_b$$

$$\int = \int_{t_a}^{t_b} \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) dt$$

We want to make time variable  $t$  complex!

It is easy to do that in zig-zag paths before continuum limit:

$$t = -i\tau \quad \varepsilon \rightarrow -i\varepsilon'$$

$$Z(b, \tau_b; a, \tau_a) = K(b, it_b; a, it_a)$$

$$= \lim_{\varepsilon' \rightarrow 0} \int dx_1 \dots dx_{N-1} \left( \frac{m}{2\pi\hbar\varepsilon'} \right)^{\frac{N}{2}} \times$$

$$\times \exp \left\{ -\frac{m}{2\hbar\varepsilon'} \sum_{i=1}^N (x_i - x_{i-1})^2 - \frac{\varepsilon'}{\hbar} \sum_{i=1}^N V\left(\frac{x_{i-1} + x_i}{2}\right) \right\}$$

well defined gaussian type integral!

$$Z = \text{Tr} Z(b, a) = \int dx_a Z(a, a)$$

This is just the  $\text{Tr} K$  continued to imaginary time

$$Z = \sum_n e^{-\frac{E_n T}{\hbar}}$$

$$\hookrightarrow \int A[x(\tau)] e^{-\frac{1}{\hbar} \int_{\tau_a}^{\tau_b} \left( \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x(\tau)) \right) d\tau}$$

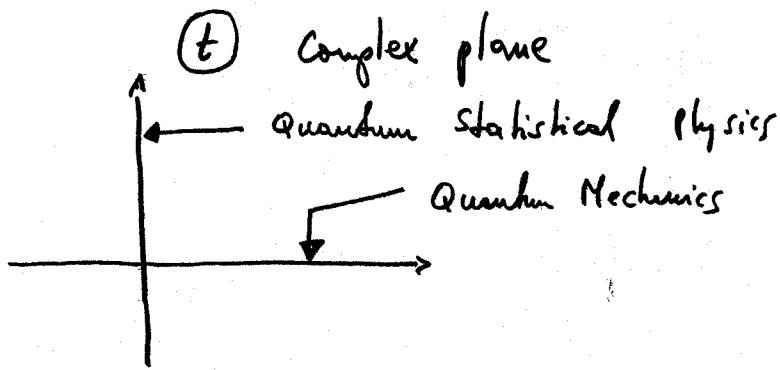
$$x(\tau_b) = x(\tau_a)$$

$$Z = \sum_n e^{-\frac{E_n}{kT}}$$

quantum partition function  
of particle in heat bath  
at temperature  $T$

$$\frac{\tau}{\hbar} \rightarrow \frac{1}{kT}$$

Imaginary time path integral for periodic paths  
is equivalent to quantum statistical physics of  
particle



Let us investigate what happened in action integral:

$$iS = i \int_{t_a}^{t_b} \left( \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x(t)) \right) dt$$

$$= - \int_{\tau_a}^{\tau_b} \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x(\tau)) \right] d\tau$$

$$t = -i\tau$$

$$\left( \frac{dx}{dt} \right)^2 = - \left( \frac{dx}{d\tau} \right)^2$$

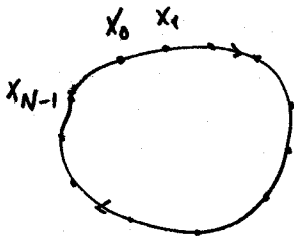
The imaginary time period for the closed paths can be chosen to relate to the heatbath temperature:

$$\tau = \frac{\hbar}{kT}$$

Imaginary time path integral has three great advantages:

- (1) It is a well-behaved real integral  
No complicated phase cancellations
- (2) Direct information on Quantum Statistical Behavior of particle at finite temperature
- (3) New analogy with a classical statistical mechanical chain lends itself to modern simulation methods

Consider a closed ring:



$x_i$  measures displacements of (an) harmonic chain from its null position

$$E = \sum_{i=1}^N \frac{1}{2\varepsilon'} m (x_i - x_{i-1})^2 + \varepsilon' \sum_{i=1}^N V\left(\frac{x_{i-1} + x_i}{2}\right)$$

$$E \rightarrow \int \left( \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + V(x(t)) \right) dt$$

in the continuum limit

$$Z = \sum_n - \frac{E_n}{kT}$$

↑  
Summation is actually an integration  
over all classical configurations

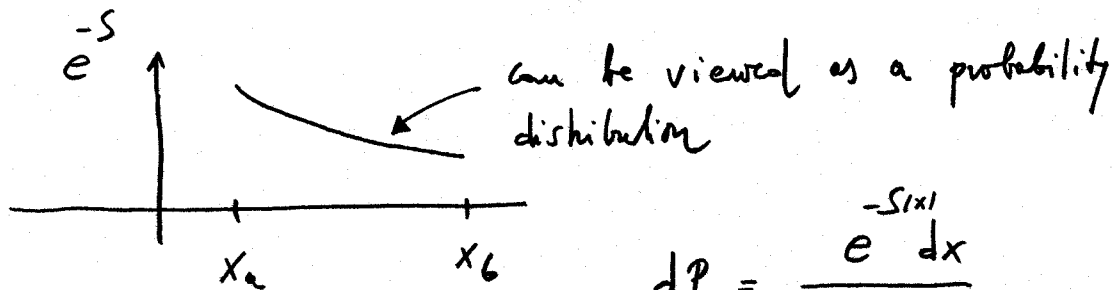
It is very much like a polymer chain

How do we integrate?

Consider the integral

$$\langle x^2 \rangle = \frac{\int_{x_a}^{x_b} x^2 e^{-S(x)} dx}{\int_{x_a}^{x_b} e^{-S} dx}$$

$S = \frac{1}{2} x^2$   
(would work for any  $S(x)$ )



$$dP = \frac{e^{-S(x)} dx}{\int_{x_a}^{x_b} e^{-S(x)} dx}$$

$$\langle x^2 \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum x_i^2$$

↑  
points distributed accordingly

Metropolis procedure:

choose  $x'$  from  $x \pm \Delta x$  interval randomly

if  $S(x') < S(x)$  accept

if  $S(x') > S(x)$  accept with  $\frac{e^{-S(x')}}{e^{-S(x)}}$  probability