PHYSICS 4E | FINAL EXAM | SPRING QUARTER 2011
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Formulas and constants:

$hc = 12{,}400\ eV\,A$; $k_B = 1/11{,}600\ eV/K$; $ke^2 = 14.4 eVA$; $m_ec^2 = 0.511\times10^6 eV$; $m_p/m_e = 1836$

Relativistic energy - momentum relation $E = \sqrt{m^2c^4 + p^2c^2}$; $c = 3\times10^8 m/s$

Photons: $E = hf$; $p = E/c$; $f = c/\lambda$ Lorentz force: $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

Planck's law : $u(\lambda) = n(\lambda)\bar{E}(\lambda)$; $n(\lambda) = \frac{8\pi}{\lambda^4}$; $\bar{E}(\lambda) = \frac{hc}{\lambda}\frac{1}{e^{hc/\lambda k_BT} - 1}$

Energy in a mode/oscillator : $E_f = nhf$; probability $P(E) \propto e^{-E/k_BT}$

Stefan's law : $R = \sigma T^4$; $\sigma = 5.67\times10^{-8} W/m^2K^4$; $R = cU/4$, $U = \int_0^\infty u(\lambda)d\lambda$

Planck : $u(\lambda) = n(\lambda)\bar{E}(\lambda)$; $n(\lambda) = \frac{8\pi}{\lambda^4}$; $\bar{E}(\lambda) = \frac{hc}{\lambda}\frac{1}{e^{hc/\lambda k_BT} - 1}$; Wien : $\lambda_m T = hc/4.96k_B$

Photoelectric effect : $eV_0 = (\frac{1}{2}mv^2)_{\max} = hf - \phi$, $\phi \equiv$ work function

Compton : $\lambda_2 - \lambda_1 = \frac{h}{m_ec}(1-\cos\theta)$; $\lambda_c \equiv \frac{h}{m_ec} = 0.0243A$; Rutherford : $b = \frac{kq_\alpha Q}{m_\alpha v^2}\cot(\theta/2)$; $\Delta N \propto \frac{1}{\sin^4(\theta/2)}$

Electrostatics : $F = \frac{kq_1q_2}{r^2}$ (force) ; $V = \frac{kq}{r}$ (potential) ; $U = q_0V$ (potential energy)

Hydrogen spectrum : $\frac{1}{\lambda} = R(\frac{1}{m^2} - \frac{1}{n^2})$; $R = 1.097\times10^7\ m^{-1} = \frac{1}{911.3A}$

Bohr atom : $r_n = r_0n^2$; $r_0 = \frac{a_0}{Z}$; $E_n = -E_0\frac{Z^2}{n^2}$; $a_0 = \frac{\hbar^2}{mke^2} = 0.529A$; $E_0 = \frac{ke^2}{2a_0} = 13.6eV$; $L = mvr = n\hbar$

$E_k = \frac{1}{2}mv^2$; $E_p = -\frac{ke^2Z}{r}$; $E = E_k + E_p$; $F = \frac{ke^2Z}{r^2} = m\frac{v^2}{r}$; $hf = hc/\lambda = E_n - E_m$

Reduced mass : $\mu = \frac{mM}{m+M}$; X - ray spectra : $f^{1/2} = A_n(Z-b)$; K : $b = 1$, L : $b = 7.4$

de Broglie : $\lambda = \frac{h}{p}$; $f = \frac{E}{h}$; $\omega = 2\pi f$; $k = \frac{2\pi}{\lambda}$; $E = \hbar\omega$; $p = \hbar k$; $E = \frac{p^2}{2m}$; $\hbar c = 1973\ eV\,A$

Wave packets : $y(x,t) = \sum_j a_j\cos(k_jx - \omega_jt)$, or $y(x,t) = \int dk\, a(k)\, e^{i(kx-\omega(k)t)}$; $\Delta k\Delta x \sim 1$; $\Delta\omega\Delta t \sim 1$

group and phase velocity : $v_g = \frac{d\omega}{dk}$; $v_p = \frac{\omega}{k}$; Heisenberg : $\Delta x\Delta p \sim \hbar$; $\Delta t\Delta E \sim \hbar$

Wave function $\Psi(x,t) = |\Psi(x,t)|e^{i\theta(x,t)}$; $P(x,t)\,dx = |\Psi(x,t)|^2\,dx$ = probability

Schrodinger equation : $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi(x,t) = i\hbar\frac{\partial\Psi}{\partial t}$; $\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$

Time – independent Schrodinger equation : $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x) = E\psi(x)$; $\int_{-\infty}^{\infty} dx\ \psi^*\psi = 1$

∞ square well : $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$; $E_n = \frac{\pi^2\hbar^2n^2}{2mL^2}$; $x_{op} = x$, $p_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$; $<A> = \int_{-\infty}^{\infty} dx\psi^*A_{op}\psi$

Eigenvalues and eigenfunctions : $A_{op}\Psi = a\Psi$ (a is a constant) ; uncertainty : $\Delta A = \sqrt{<A^2> - <A>^2}$

Harmonic oscillator: $\Psi_n(x) = C_n H_n(x) e^{-\frac{m\omega}{2\hbar}x^2}$; $E_n = (n+\frac{1}{2})\hbar\omega$; $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$; $\Delta n = \pm 1$

Step potential: $R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$, $T = 1 - R$; $k = \sqrt{\frac{2m}{\hbar^2}(E - V)}$

Tunneling: $\psi(x) \sim e^{-\alpha x}$; $T \sim e^{-2\alpha\Delta x}$; $T \sim e^{-2\int_a^b \alpha(x)dx}$; $\alpha(x) = \sqrt{\frac{2m[V(x) - E]}{\hbar^2}}$

3D square well: $\Psi(x,y,z) = \Psi_1(x)\Psi_2(y)\Psi_3(z)$; $E = \frac{\pi^2\hbar^2}{2m}(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2})$

Spherically symmetric potential: $\Psi_{n,\ell,m}(r,\theta,\phi) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$; $Y_{\ell m}(\theta,\phi) = f_{lm}(\theta)e^{im\phi}$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$; $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$; $L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m}$; $L_z = m\hbar$

Radial probability density: $P(r) = r^2 |R_{n,\ell}(r)|^2$; Energy: $E_n = -13.6eV\frac{Z^2}{n^2}$

Ground state of hydrogen and hydrogen-like ions: $\Psi_{1,0,0} = \frac{1}{\pi^{1/2}}(\frac{Z}{a_0})^{3/2} e^{-Zr/a_0}$

Orbital magnetic moment: $\vec{\mu} = \frac{-e}{2m_e}\vec{L}$; $\mu_z = -\mu_B m_l$; $\mu_B = \frac{e\hbar}{2m_e} = 5.79 \times 10^{-5} eV/T$

Spin 1/2: $s = \frac{1}{2}$, $|S| = \sqrt{s(s+1)}\hbar$; $S_z = m_s\hbar$; $m_s = \pm 1/2$; $\vec{\mu}_s = \frac{-e}{2m_e}g\vec{S}$

Total angular momentum: $\vec{J} = \vec{L} + \vec{S}$; $|J| = \sqrt{j(j+1)}\hbar$; $|l - s| \le j \le l + s$; $-j \le m_j \le j$

Orbital+spin mag moment: $\vec{\mu} = \frac{-e}{2m}(\vec{L} + g\vec{S})$; Energy in mag. field: $U = -\vec{\mu}\cdot\vec{B}$

Two particles : $\Psi(x_1,x_2) = +/- \Psi(x_2,x_1)$; symmetric/antisymmetric

Screening in multielectron atoms: $Z \rightarrow Z_{eff}$, $1 < Z_{eff} < Z$

Orbital ordering:

1s < 2s < 2p < 3s < 3p < 4s < 3d < 4p < 5s < 4d < 5p < 6s < 4f < 5d < 6p < 7s < 6d ~ 5f

$f_B(E) = Ce^{-E/kT}$; $f_{BE}(E) = \frac{1}{e^{\alpha}e^{E/kT} - 1}$; $f_{FD}(E) = \frac{1}{e^{\alpha}e^{E/kT} + 1}$; $n(E) = g(E)f(E)$

Rotation: $E_R = \frac{L^2}{2I}$, $I = \mu R^2$, vibration: $E_v = \hbar\omega(v + \frac{1}{2})$, $\omega = \sqrt{K/\mu}$, $\mu = m_1 m_2/(m_1 + m_2)$

$g(E) = [2\pi(2m)^{3/2}V/h^3]E^{1/2}$ (translation, per spin) ; Equipartition: $<E> = k_B T/2$ per degree of freedom

Einstein coefs.: $B_{12} = B_{21}$, $A_{21}/(B_{21}u(f)) = e^{hf/kT} - 1$; $u(f) = (8\pi hf^3/c^3)/(e^{hf/kT} - 1)$

Justify all your answers to all (7) problems

Problem 1 (10 pts)

Radiation in a cavity is in thermal equilibrium. The radiation emitted through a small hole in one of the walls has maximum power at wavelength $\lambda = 4000A$.
(a) What is the temperature of this cavity?
(b) What is the energy of each photon in the cavity with wavelength $\lambda = 4000A$? What is it for each photon in the cavity with wavelength $\lambda = 40{,}000A$?
(c) The average number of photons in a mode of wavelength λ at temperature T in this cavity is $1/(e^{\frac{hc}{\lambda kT}} - 1)$. Show that there are many more photons in this cavity in each mode of wavelength 40,000A than there are in each mode of wavelength 4000A. How many times more?
(d) Given the answer in (c), explain why it is that this body emits more power at wavelength 4000A than at wavelength 40,000A. First give a qualitative answer, then calculate exactly how much more power does it emit at 4000A than at 40,000A.

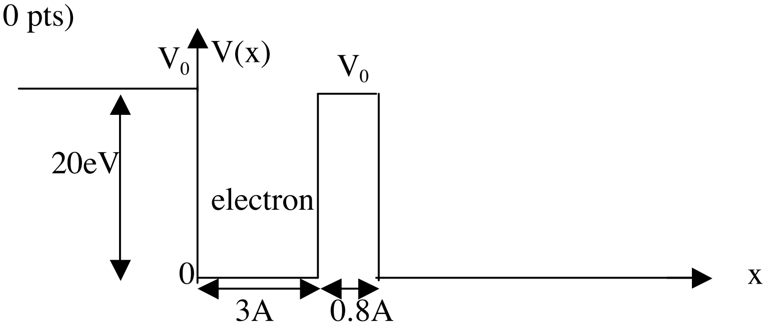
Problem 2 (10 pts)

When radiation in the wavelength range $100A \leq \lambda \leq 200A$ is incident on a gas of hydrogen-like ions at room temperature, two absorption lines are seen and three emission lines are seen.
(a) What is the Z (atomic number) of these ions?
(b) What are the wavelengths of these absorption and emission lines, in A?
(c) What is the speed of the electron in the ground state of these ions according to the Bohr model? Give your answer as v/c.

Problem 3 (10 pts)

An electron and a neutron are in the same one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}Kx^2$, with K a constant. Assume for the purposes of this problem that the ratio mass of neutron / mass of electron = 1849. The electron is in its ground state, and the neutron is in a state that has the same energy as that of the electron. The classical amplitude of oscillation for the electron in the state it's in is 1A.
(Note: classical amplitude is defined as the amplitude of oscillation for a classical particle that has the same energy as the quantum oscillator).
(a) How many energy states are there for the neutron with energy less than the one it is in? Give the quantum numbers of these states.
(b) Suppose you take snapshots showing the positions of the electron and the neutron. Where is the electron most likely to be found, and where is the neutron most likely to be found? Give the values of the coordinate x for each, in A. Justify it.
(c) Suppose the neutron makes transitions and ends up in its ground state. What is the ground state energy of the neutron compared to that of the electron? Give their ratio.
(d) For the electron and the neutron in their respective ground states in this potential, which has larger uncertainty in its position? Calculate the ratio of the uncertainties in the positions of the neutron and the electron. Hint: one way to find the answer is to argue that the uncertainties are of order the classical amplitudes of oscillation. There are other ways.

Problem 4 (10 pts)



An electron is in the potential well shown in the figure, of depth V_0=20eV and width 3A.
(a) Find the energy of the ground state and the first excited state. You may assume it is an infinite well and get approximate answers. State whether the exact answers will be larger or smaller than the approximate answer, and for which of the two states (ground state or first excited state) is the error larger and why.
(b) For the electron in the first excited state it takes on average a time t_0 to tunnel through the potential barrier of thickness 0.8A and escape in the +x direction. How long will it take on average if the electron is instead in the ground state? Give your answer in terms of t_0.
Hint: you should take into account that the attempt frequency to escape is different in both states because of the different speeds of the electron.

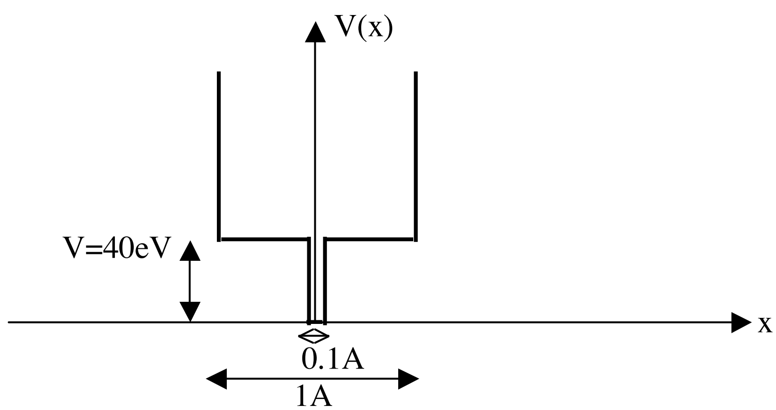
Problem 5 (10 pts)

The wave function for an electron in a hydrogen-like ion is
$\psi(r,\vartheta,\phi) = Cr^2 e^{-r/a_0} \sin\vartheta \cos\vartheta e^{-2i\phi}$
with C a constant.
(a) Give the values of the quantum numbers n, ℓ, m_ℓ and of the atomic number Z.
(b) Find the most probable value for r.
(c) Compare the value found in (b) with the prediction of the Bohr atom for the orbit with this value of n.
(d) Find the change in energy of this electron when a magnetic field of magnitude B=7T in the +z direction is turned on, ignoring spin. Does it increase or decrease?
(e) Same as (d) if instead the magnetic field is pointing in the +x direction.

Problem 6 (10 pts)

As you know the characteristic temperature for rotation of a diatomic molecule is
$T_R = \frac{\hbar^2}{2I\,k}$ with I the moment of inertia and k Boltzmann's constant.
(a) Give an estimate for the average rotational energy of a diatomic molecule at temperature $T=10T_R$. Give the answer in terms of T_R.
(b) Calculate an estimate for the average rotational energy of a diatomic molecule at temperature $T=0.1T_R$. Give the answer in terms of T_R. How much smaller or larger is it than the result in (a)? Hints: Use the Boltzmann distribution; take into account only the ground state and the first excited states for rotation, since the contribution from higher states is negligible at this temperature; don't forget to include the degeneracy of the states.
(c) Find for which temperature is the molecule equally likely to be in the ground state as it is to be in one of the first excited states. Give your answer in terms of T_R.
Hint: the degeneracy of the state with quantum number ℓ is the same as for the electron in such a state of hydrogen.

Problem 7 (for extra credit, 10 pts)



A particle moves in the one-dimensional potential well V(x) shown in the figure. The potential is $V=\infty$ for |x|>0.5A, V=40eV for 0.05A<|x|<0.5A and V=0 for -0.05A<x<0.05A. It is difficult to solve the Schrodinger equation exactly for this potential, but you should be able to give approximate answers to the following questions.

(a) Estimate for what range of values of the mass of the particle (in MeV/c^2) will the ground state energy of the particle (measured from the bottom of the narrow well) be less than 40eV.
(b) If the particle is an electron, estimate the value of its ground state energy, in eV, measured from the bottom of the narrow well. Justify your answer.
(c) Explain in words why it is impossible to have an electron localized inside the narrow well, i.e. in the region |x|<0.05A, for this potential V(x). Draw another potential V(x) that would cause an electron to be localized in a region of width 0.1A.
(d) Estimate roughly the value of the mass of a particle (in MeV/c^2) for which the probability that it will be found in the range |x|<0.05A in the potential V(x) shown in the figure is approximately 50%.