Final solutions

Problem 1
(a) We can write

$$S_i S_j = -m^2 + m(S_i + S_j) + \delta S_i \delta S_j$$

The last term is due to fluctuations. We then have the mean field Hamiltonian

$$H_{MF} = \frac{1}{2} N z J m^2 - (zJm + B) \sum_i S_i$$

(b) The partition function is

$$Z = e^{-\beta F} = e^{-\frac{1}{2}\beta N z J m^2} \left(\sum_{S=-1,0,+1} e^{\beta(zJm+B)S} \right)^N$$

from which we get

$$F = \frac{1}{2} N z J m^2 - N k_B T \ln \left[1 + 2\cosh\left(\frac{zJm + B}{k_B T} \right) \right].$$

In terms of dimensionless quantities we get

$$f = \frac{1}{2} m^2 - \theta \ln \left[1 + 2\cosh\left(\frac{m+h}{\theta} \right) \right].$$

Problem 2
The first condition gives

$$\left(\frac{\partial P}{\partial v} \right)_T = -\frac{RT}{(v-b)^2} + \frac{3a}{v^4} = 0$$

which is equivalent to

$$RT = 3a \frac{(v-b)^2}{v^4}.$$

The second condition gives

$$\left(\frac{\partial^2 P}{\partial v^2} \right)_T = \frac{2RT}{(v-b)^3} - \frac{12a}{v^5} = 0$$

or, equivalently

$$RT = 6a\frac{(v-b)^3}{v^5}.$$

Equating the two equations for RT we get $v_c = 2b$. Substituting this result in anyone of the equations for RT we obtain

$$T_c = \frac{3a}{16b^2R}.$$

Substituting v_c and T_c in the equation of state we get

$$P_c = \frac{a}{16b^3}.$$

Finally

$$\frac{RT_c}{P_cv_c} = \frac{3}{2}$$

which is a very low number.

Problem 3

As explained in class, we can assume that the chemical potential is zero in the region of Bose-Einstein condensation. The number of particles in the excited state is thus

$$N_{exc} = \int_0^\infty \bar{N}_\epsilon g(\epsilon)d\epsilon,$$

where

$$\bar{N}_\epsilon = \frac{1}{e^{\epsilon/k_BT} - 1},$$

and

$$g(\epsilon)d\epsilon = \frac{V4\pi p^2 dp}{h^3} = \frac{V4\pi}{h^3}\left(\frac{\epsilon}{A}\right)^{2/s}\frac{1}{s}\left(\frac{\epsilon}{A}\right)^{1/s-1} d\epsilon$$

so that

$$N_{exc} = const \times V\int_0^\infty \frac{\epsilon^{3/s-1}}{e^{\epsilon/k_BT} - 1}d\epsilon,$$

If we set $x = \epsilon/k_BT$, we get

$$N_{exc} = const' \times V(k_BT)^{3/s}.$$

Therefore, since T_B is determined by the condition $N_{exc} = N$ we find that

$$T_B \propto \left(\frac{N}{V}\right)^{s/3}.$$

The fraction

$$\frac{N_{gs}}{N} = 1 - \left(\frac{T}{T_B}\right)^{3/s}.$$

Finally, we find that

$$U = \int_0^\infty \epsilon \bar{N}_\epsilon g(\epsilon) d\epsilon \propto T^{3/s+1},$$

from which we find

$$C_V \propto T^{3/s},$$

and

$$S = \int_0^T \frac{C_V dT}{T} \propto T^{3/s}.$$