

Physics 161: Black Holes: Lectures 1 and 2: 3 and 5

Jan 2011

Professor: Kim Griest

January 5, 2011

1 Introduction

This is Physics 161, Black Holes. This course is not a prerequisite for anything, so I am assuming everyone is taking it for interest. This also means that we can modify the content to what you want to learn about. So be sure to communicate to me what you are liking and not liking. I hope this course will be a fun application of your physics, math, and engineering skills, applied to bizarre situations that really occur in nature. It's a big Universe out there and the kind of things going on are pretty amazing. I also find it amazing that we humans – dots on a tiny planet orbiting one of 100 billion stars in one of 100 billion galaxies – can actually work out what the Universe is like. You will hopefully find that the tools needed to do this are within your grasp. Astrophysics cuts across all disciplines, so you will be using much of what you learned previously. The prerequisites for this course are the entire Physics 2 or Physics 4 sequence. Since we will be studying Einstein's Theory of General Relativity (GR) and this is based upon the simpler Special Relativity, it is crucial that you have studied Special Relativity previously. For example, if you only took Physics 2A,B,C and missed out 2D, then you shouldn't take this course.

GR was Einstein's crowning achievement and in my opinion is one of the greatest achievements humans have yet produced. It is almost never taught at the undergraduate level, so there is really no adequate book. I've developed a way to get to the essential physics without using graduate level math, but we will stretch your math skills a little. I will teach the extra math you need, so don't worry.

My plan for the course is to do a mixture of hand waving explanations, analogies, examples, and mathematical calculations, some of which will be optional for you to learn. So the level of presentation will vary. For this reason interaction, questions and feedback from you will be essential. If you aren't following something, please just raise your hand and say so. GR and Black holes (BH) will definitely blow your mind, and you will discover that many "dumb" questions have very subtle answers. So be sure to ask all the dumb questions you come up with. You can and will understand black holes, curved spacetime, etc. by the end of the quarter.

The grading of this course will be a little different. There will not be any tests. There will be graded homeworks which will account for 60% of the grade. The other 40% will be from a final paper or talk given during the final. Everyone must attend the final and hear the talks. There is a handout that tells about the grading. Note that I am very concerned about academic integrity. You may not copy anything from anybody else, or from any source whatsoever. You must do all the work yourself. You may talk to others on how to do homework problems, but you must work out 100% of everything yourself; otherwise

you are cheating and if caught you will suffer very severe consequences. Please read and reread the handout for details of the grading what is expected.

Finally, note that I have not been able to find an appropriate book for this class. I'm on my 3rd book this quarter. The one I used last time was too easy; I've used the current required textbook before, but it may be somewhat too hard. So it is essential that you come to class. That is where you will learn at the level needed to do the homework.

1.1 Tour of the Universe

We will start with a slide show of the Universe. This is so that you get oriented to what is out there in the Universe, and so when I say the word “galaxy” or “supernova” you have a feeling for what I am talking about.

1.2 Black Holes

Black Holes and the expansion of the Universe (covered next quarter in Physics 162) are two subjects that rest completely on Einstein's General Relativity. We will not be able to cover GR in depth, but we will understand the essential concepts at a level even most PhD physicists do not. And we will do actual calculations of what happens around and even inside black holes. Most physicists don't study GR because it only differs from Newton's gravity theory and from Special Relativity in a few cases. But GR is Nature's choice – whenever GR differs from Newton, GR has shown to be right. It is how Nature actually works, and requires a radical rethinking of physical reality. GR and Quantum Mechanics are the two subjects I know that are most likely to surprise you.

1.3 Curved Spacetime

GR says that gravity is not really a “force”; but instead is curved spacetime. What does that mean?

Galileo and Newton view motion with respect to a rigid Euclidean reference frame that extends throughout all space and endures forever. Within this ideal frame, there exists the mysterious force of gravity – a foreign influence. Einstein says, “there is no such thing”. Climb into a spaceship and see for yourself – no gravity there! Suppose you are floating in a space ship with no windows. Can you tell you whether you are out in the middle of free space or orbiting the Earth? Not really!

This is the starting point of GR, *Physics is locally gravity free*. All free particles move in straight lines and constant speed. In an inertial frame, physics looks simple. But such frames are inertial only in a limited region, i.e. *local*. Complications arise when motion is described in nearby local frames. Any difference between direction in one local frame and a nearby frame is described in terms of “curvature of spacetime”. Curvature implies it is impossible to use a single Euclidean frame for all space. In a small region, curvature is small, that is it looks flat. Einstein adds together many local regions and has a theory with no gravity force. Newton has a single flat space and an extra force. These are radically different views. Einstein is right, but usually Newton's view is good enough for calculation.

1.4 Example of two surveyors

Fig 1: Surveyors on Earth going north.

Let me give an example that is extremely helpful in understanding what I just said. Consider two surveyors standing 100 meters apart on the equator. They both decide to start out perfectly parallel

towards the north by rolling a big ball directly north. Some time later as they roll their balls, one notices that the distance between the two balls is less than the initial 100m. “Hey” one surveyor calls, you aren’t going straight, you are coming towards me. The other says “I’m going perfectly straight, it’s you that’s moving.” After a lot of checking they decide they both are rolling the balls straight, but that there must be some mysterious force that is pulling the balls toward each other. (What is happening of course, is that both balls are approaching the north pole, and would hit each other there.) They try the experiment with bigger balls and discover that the big balls come closer as they go north by the same amount. Since $F = ma$, the bigger balls require a bigger force and thus they decide this force is proportional to the mass of the object. In fact, it seems all objects moving north attract all other objects with a force proportional to their mass. “We have made a great discovery; let’s call this force gravity”, the surveyors decide.

The surveyors think they have a new force because they think they are moving on a flat surface, but in reality are on the large curved surface of the Earth. They don’t realize the reason for the balls coming together is the curvature of the Earth’s surface. In fact, you can do the math for the radius of the Earth and even find the value of the effective “Newton’s constant G ” (not the same of course as our normal G , and this “gravity” does not fall-off as r^{-2} .)

From Einstein’s view, there is no force. The movement together of the balls is proof that the Earth’s surface is curved. Einstein says the same thing with regard to actual gravity that pulls the falling apple toward the Earth. No force, but curved spacetime. Note in the example of the surveyors only space (Earth surface) was curved; in GR both space and time are curved. This view in fact explains a major mystery of Newton’s law. Newton had two types of mass: $m = F/a$ is “inertial mass”, telling how hard it is to accelerate things, while the m in $F = GMm/r^2$ is the gravitational mass, telling how much gravity comes off the object. Why are these masses the same? In Coulomb’s law, the source of the force is the charge, and it is not the same as the mass. This is a mystery, but it has been tested carefully many times and the two masses are always equal. Einstein’s answer is that there is only the inertial mass, which curves spacetime. Gravity as a force, doesn’t exist.

1.5 Tidal force as curvature

The principle of relativity you learned in Special Relativity says physics is the same in all inertial frames. Consider traveling in a moving train or plane. Drop a ball; it falls just like when standing on the ground. You can play catch or pour wine on a plane, even though for someone watching from outside the plane the ball or wine would travel in a parabola. The principle of relativity says one cannot tell whether or not one is moving in a frame with constant velocity (except by looking outside at someone else). So consider a mass floating in an orbiting rocket ship; not touching anything, just floating. Where does it get its marching orders from? Newton says both the mass and ship get their orders from the distant Earth. Einstein says the mass gets its orders locally. A free falling frame is a “local inertial frame” so since there is nothing inside the spaceship pushing on the mass, it stays still with respect to the spaceship. In fact, according to Einstein both the space ship and mass are sampling the local curvature of spacetime which is what is causing them to orbit. Things move in “straight lines” in inertial frames; the mass can veer, but only responding to structure of spacetime right there. Newton says the mass would go “straight” in his ideal all pervading reference frame but the Earth deflects it.

How do you tell if a frame is inertial? Easy, just check every particle, light ray, etc. to see if they move in straight lines at constant speed. So inside the space ship it is an inertial frame and everything moves simply. Simple? Too simple! Where is gravity at all? How do we see the curvature?

Fig. 2: Balls in a space ship

Consider two balls in a space ship. We put them side by side 25 m apart. If the space ship is in orbit, the balls just float there. They don't move apart or together, and if there were no windows, there would be no way to tell they were in orbit above the Earth or in the middle of space far from any star or planet. Now, instead of in orbit, drop the entire space ship from a height of 250 meters above the earth. The ship and balls both fall straight down, and will hit the ground 7 seconds later ($t = \sqrt{2d/g}$). While falling, the balls still seem to be floating in deep space away from all forces. However, if you check carefully there is a small effect. Going straight down towards the Earth's center, the balls are about 1×10^{-3} m closer together when they hit. $l = \theta r$, $dl/l = dr/r$, $dl = l dr/r = (25)(250)/6.4 \times 10^6 = 1 \times 10^{-3}$ m. Watching this from the ground it is clear what is happening, but inside it seems as if the balls are attracting each other. After 7 seconds they have moved about 1 mm closer. This is not actually the gravity attraction between the two balls, but is the "tidal" force and in fact proves that the space is curved. Note that if your measuring instruments had an accuracy of worse than 1 mm, then this attraction could not be detected. We say that to an accuracy of 1 mm and a time under 7 seconds this 25 m wide space is a local inertial frame, but for longer times, or better accuracy, it is not. Smaller size ships and shorter times give more approximately inertial frames. However, if you add enough small frames together you can detect the curvature. Consider a ring of balls above the Earth's surface each separated by 25 m and drop them all together. After 7 seconds they are all 1 mm closer. In each frame you can't see it, but by adding up all the frames you see that entire circle around the Earth has shrunk. The factor is $1 \text{ mm}/25000 \text{ mm} = 1/25000$, and the distance from the center of the Earth shrinks by same factor $(1/25000)(6.4 \times 10^6)m = 250m$. Note this is just like the distance around a line of latitude shrinks for surveyors rolling balls toward north pole. The smallness of this effect in a single spaceship actually shows the smallness of the curvature of spacetime, which is part of the reason GR is not easy to experimentally distinguish from Newton.

1.6 The metric

In GR the key concept is the metric. GR replaces gravity with curvature of spacetime. The metric tells how to measure distances in space and time. The metric contains all the info about curvature in a simple formula. It is the key to understanding GR and to be able to calculate anything.

*Examples of metrics*¹

- 3-D flat space metric: $ds = \sqrt{dx^2 + dy^2 + dz^2}$, (or $ds^2 = dx^2 + dy^2 + dz^2$). This is just the Pythagorean theorem! (We use dx rather than x because we want to talk about "local" curvature. We find x from dx by integrating.)
- 2-D flat space metric: $ds^2 = dx^2 + dy^2$.
- 4-D flat space: $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$. w is "4th" dimension; note how easy this is to write mathematically, but it is hard to visualize! Note the number of dimensions described by a metric is the number of different differentials.
- 2-D curved space metric (surface of sphere):

Fig. 3: Coordinates for surface of sphere

¹Technically the metric is a rank two tensor, e.g. a matrix, and what we are showing here is called the "line element". We will use the terms "metric" and "line element" interchangeably in this course, since they contain the same information, but in a more advanced course the distinction would be made.

$ds^2 = r_0^2 d\theta^2 + r_0^2 \sin^2 \theta d\phi^2$, where r_0 is a constant. Note if you define $dx' = r_0 d\theta$ and $dy' = r_0 \sin \theta d\phi$ then *locally* $ds^2 = dx'^2 + dy'^2$, and it looks like flat space in small enough areas. However, if you move far then θ changes, and the distance between points on a sphere is *not* given by a flat space formula: $s \neq \sqrt{(x_0 - x')^2 + (y_0 - y')^2}$. You need to do the integral:

$$s = \int_{\theta_0, \phi_0}^{\theta_1, \phi_1} ds = \int_{\theta_0, \phi_0}^{\theta_1, \phi_1} \sqrt{r_0^2 d\theta^2 + r_0^2 \sin^2 \theta d\phi^2}.$$

This is a longer distance than the flat space distance between these points.

- 3-D curved metric. Above we had a curvature in 2-D space. How about curved 3-D space? Easy to write mathematically, but hard to visualize. For example, $ds^2 = \sqrt{x^2 + y^2 + z^2} dx^2 + dy^2 + dz^2$ is a curved 3-D space. Just add almost any mathematical function to the flat 3-D metric and

it will be curved. We can try to visualize using “embedding”. In the curved 2-D metric example we can visualize the curvature by drawing a 3-D sphere, that is by curving the 2-D surface through 3-D space. Similarly we could visualize curved 3-D space as curved through a 4-D space! But note the curvature of 3-D space does not need the 4th dimension. That sounds hard, but if we can ignore one of the 3-D spatial dimensions, we again have curved 2-D embedded in 3-D. We will show examples later. Note that the space around the Earth is curved, but we can’t see that curvature! We can measure its effects however and prove it is there! It becomes an experimental question.

1.7 Using a metric to find distances, areas, etc.

The left hand side of a metric (aka line element) gives the real distance measured in meters. The right hand side gives the coordinates, specified by the differentials, as well as functions and constants. The coordinates may or may not be in units of distance, so to find an actual distance, area, or volume, the left hand side must be used. For example, consider the 3-D metric

$$ds^2 = f(x, y, z) dx^2 + g(x, y, z) dy^2 + h(x, y, z) dz^2,$$

where $f(x, y, z)$, $g(x, y, z)$, and $h(x, y, z)$ are functions that specify the curvature of the space. To find a distance in the x direction, one would hold y and z constant. That is one would set $dy = 0$ and $dz = 0$, to get $ds \equiv dl_x = \sqrt{f} dx$. The actual distance from x_0 to x_1 would therefore be $l_x = \int_{x_0}^{x_1} \sqrt{f} dx$. Likewise to measure a distance in the z direction you would set $dx = dy = 0$ and integrate $dl_z = \sqrt{h} dz$. The distance along an arbitrary curve can be found by factoring out, say dx , and integrating: $l = \int dl = \int \sqrt{f + g(dy/dx)^2 + h(dz/dx)^2} dx$, where dy/dx and dz/dx are calculated along the integration path.

Finally, while in flat space a small area is given by $dA = dx dy$, in curved space one must use $dA = dl_x dl_y = \sqrt{fg} dx dy$ (or $dl_y dl_z$, etc.). The volume element is thus $dV = dl_x dl_y dl_z = \sqrt{fgh} dx dy dz$.