Physics 161: Black Holes: Lecture 14: 4 Feb 2011

Professor: Kim Griest

14 Geodesics and motion of light around a black hole

In one sense light is just the same as a particle with it's rest mass $m \to 0$. To find the geodesics for light, then, we should be able to just take the limit $m \to 0$ in the geodesic equations we already found. As a reminder I reproduce those equations below, in the special case with $\theta = \pi/2$ relevant for motion in the x-y plane.

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m} - \left(1 - \frac{r_S}{r}\right)\left(m + \frac{l^2}{mr^2}\right)$$

$$\frac{d\phi}{d\tau} = \frac{l}{mr^2}$$

$$\frac{d\theta}{d\tau} = 0$$

$$\frac{dt}{d\tau} = \frac{E}{m}\left(1 - \frac{r_S}{r}\right)^{-1}.$$

But we immediately see that if we try to take the limit $m \to 0$ in these equations we get into trouble. We get nonsense like $0 = \infty$, etc. Actually we talked about this trouble before when discussing light. For light the proper time $d\tau$ is always zero. Thus when we chose to use τ for our affine parameter, we excluded the use of these equations for light! What we should do is go back to the Schwarzschild metric, choose a different affine paramter, and use the Euler-Lagrange equations to rederive the geodesics for light. However, we can instead use a trick. This is another of those useful tricks worth learning.

We note that in the limit $m \to 0$, the speed of the particle goes to c and relativistic factor $\gamma \to \infty$. That is why the proper time $\tau \to 0$. However, the combination $m\gamma$ is the energy of the particle and stays fine as $m \to 0$. Thus if we use $\lambda = \tau/m$ as the affine parameter everything may work out fine. We do that by substituting $\tau = m\lambda$ everywhere in the geodesic equations above, and then multiplying through by m:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{r_S}{r}\right)\left(m^2 + \frac{l^2}{r^2}\right)$$

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}$$

$$\frac{d\theta}{d\lambda} = 0$$

$$\frac{dt}{d\lambda} = E\left(1 - \frac{r_S}{r}\right)^{-1}.$$

Now we can take the limit $m \to 0$ and get equations that make sense. Only the r equation changes to

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{r_S}{r}\right)\frac{l^2}{r^2}.$$

Note that the trick we did was to try to rescale our original equation that had a bad affine parameter τ to a different affine parameter that behaved ok. We scaled by a constant, such that we could take the limit we wanted. If the equations we got didn't make sense we might have tried another rescaling. The basic idea of rescaling differential equations is a trick that is often useful when you want to take some limit of the equations.

Note one problem with the above is that we can't use the old definitions of E and l which involve m. We know from the Euler-Lagrange treatment that these quantities are conserved along geodesics because there is no explicit t or ϕ dependence in the metric, so we can still call them energy and angular momentum, but we don't know yet what the formulas for them are when considering light geodesics.

OK, we are now ready to do the effective potential treatment for the photon geodesic equation. We count $(dr/d\lambda)^2$ as the kinetic energy term, and therefore find that

$$V_{eff} = \left(1 - \frac{r_S}{r}\right)\frac{l^2}{r^2} = \frac{l^2}{r^2} - \frac{r_S l^2}{r^3}.$$

Plotting this we can discover the light orbits around a black hole.

Fig: Effective potential for light orbits in Schwarzschild metric

Using what we learned from the massive case we see that there are three types of orbits: an unstable circular orbit, a coming in, then out orbit, and a capture orbit. The really new thing here is the circular light orbit. In Newtonian mechanics light cannot orbit anything, but here in GR it can! Let's find the radius of the light orbit:

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{r^3} + \frac{3r_S l^2}{r^4} = 0,$$

or

$$r_{lightorbit} = \frac{3}{2} r_S.$$

Thus at $1.5r_S$, just half the last stable orbit for massive particles, light will orbit the hole. Since it is an unstable orbit, a black hole can trap light for some amount of time, and then release it back out to infinity.

Light orbits can produce some pretty wierd effects. Suppose an advanced race built a tube around a black hole just at the light orbit radius. This is outside the trapped surface, so people could go in that tube and still get away from the black hole. If you were standing in such a tunnel, what would you see? The light would go round and round, and you would see the back of your head. Actually further down the tube you would see yourself again. The tube would in fact look infinitely long, and perfectly straight. If you tried to measure its length with a laser you would never find the end. So by looking you would not know that you were in a circular tube! It would look perfectly straight to you!

Even weirder would be if you built a such a tube a tiny bit inside the light orbit radius $r = 1.5r_S$. In this case if you shone your laser beam, it would curve inward towards the hole and hit the inner wall of the tube. Thus to you it would seem that the inner wall was further out than the center. The appearance would be that the entire tube was curving outward, not inward! However, if you got on your skate board and gave yourself a push, you it would not take any energy to coast around the hole, it is all at the same

r. It would look like you were going uphill, but feel like you were going level! Finally, what would it be like if you built the tube just outside the light orbit radius? Check out the effective potential; light trying to circle just outside the peak in the potential (light orbit radius) will “fall” out to infinity. Thus the light would go for a way down the tube but eventually hit the ceiling. Thus it would look like the tube case curing down towards the hole (which it is). However, again a skateboard ride would feel like you are going straight, even though it looks like you are going downhill.