5-25 To find the energy width of the $\gamma$-ray use $\Delta E \Delta t \geq \frac{\hbar}{2}$ or

$$\Delta E \geq \frac{\hbar}{2\Delta t} \geq \frac{6.58 \times 10^{-16}\ \text{eV}\cdot\text{s}}{(2)(0.10 \times 10^{-9}\ \text{s})} \geq 3.29 \times 10^{-6}\ \text{eV}.$$

As the intrinsic energy width of $\sim \pm 3 \times 10^{-6}\ \text{eV}$ is so much less than the experimental resolution of $\pm 5\ \text{eV}$, the intrinsic width can't be measured using this method.

5-26 The full width at half-maximum (FWHM) is 110 MeV. So $\Delta E = 55$ MeV and using $\Delta E_{\text{min}} \Delta t_{\text{min}} = \frac{\hbar}{2}$,

$$\Delta t_{\text{min}} = \frac{\hbar}{2\Delta E} = \frac{6.58 \times 10^{-16}\ \text{eV}\cdot\text{s}}{2(55 \times 10^{6}\ \text{eV})} \cong 6.0 \times 10^{-24}\ \text{s}$$

$$\tau = \text{lifetime} \sim 2\Delta t_{\text{min}} = 1.2 \times 10^{-23}\ \text{s}$$

5-27 For a single slit with width a, minima are given by $\sin\theta = \frac{n\lambda}{a}$ where $n = 1, 2, 3, \ldots$ and $\sin\theta \approx \tan\theta = \frac{x}{L}$, $\frac{x_1}{L} = \frac{\lambda}{a}$ and $\frac{x_2}{L} = \frac{2\lambda}{a} \Rightarrow \frac{x_2 - x_1}{L} = \frac{\lambda}{a}$ or

$$\lambda = \frac{a\Delta x}{L} = \frac{5\ \text{Å} \times 2.1\ \text{cm}}{20\ \text{cm}} = 0.525\ \text{Å}$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{(1.24 \times 10^4\ \text{eV}\cdot\text{Å})^2}{2(5.11 \times 10^5\ \text{eV})(0.525\ \text{Å})^2} = 546\ \text{eV}$$

5-29 With *one* slit open $P_1 = |\Psi_1|^2$ or $P_2 = |\Psi_2|^2$. With <u>both</u> slits open, $P = |\Psi_1 + \Psi_2|^2$. At a maximum, the wavefunctions are in phase so

$$P_{\text{max}} = (|\Psi_1| + |\Psi_2|)^2.$$

At a minimum, the wavefunctions are out of phase and

$$P_{\text{min}} = (|\Psi_1| - |\Psi_2|)^2.$$

Now $\frac{P_1}{P_2} = \frac{|\Psi_1|^2}{|\Psi_2|^2} = 25$ or $\frac{|\Psi_1|}{|\Psi_2|} = 5$, and

$$\frac{P_{\text{max}}}{P_{\text{min}}} = \frac{(|\Psi_1| + |\Psi_2|)^2}{(|\Psi_1| - |\Psi_2|)^2} = \frac{(5|\Psi_2| + |\Psi_2|)^2}{(5|\Psi_1| - |\Psi_2|)^2} = \frac{6^2}{4^2} = \frac{36}{16} = 2.25.$$

5-32 (a) $$f = \frac{E}{h} = \frac{(1.8)(1.6 \times 10^{-19}\ \text{J})}{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}} = 4.34 \times 10^{14}\ \text{Hz}$$

(b) $\lambda = \frac{c}{f} = 691\text{ nm}$

(c) $\Delta E \geq \frac{\hbar}{\Delta t} = \frac{6.63 \times 10^{-34}\text{ J}\cdot\text{s}}{2\pi\left(2 \times 10^{-6}\text{ s}\right)}$

$\Delta E \geq 5.276 \times 10^{-29}\text{ J} = 3.30 \times 10^{-10}\text{ eV}$

6-2 (a) Normalization requires

$$1 = \int_{-\infty}^{\infty} |\psi|^2\, dx = A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{A^2}{2}\right) \int_{-\frac{L}{4}}^{\frac{L}{4}} \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right) dx$$

so $A = \frac{2}{\sqrt{L}}$.

(b) $$P = \int_0^{\frac{L}{8}} |\psi|^2\, dx = A^2 \int_0^{\frac{L}{8}} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{4}{L}\right)\left(\frac{1}{2}\right) \int_0^{\frac{L}{8}} \left(1 + \cos\left(\frac{4\pi x}{L}\right) dx\right)$$

$$= \left(\frac{2}{L}\right)\left(\frac{L}{8}\right) + \left(\frac{2}{L}\right)\left(\frac{L}{4\pi}\right) \sin\left(\frac{4\pi x}{L}\right) \Bigg|_0^{\frac{L}{8}} = \frac{1}{4} + \frac{1}{2\pi} = 0.409$$

6-3 (a) $A \sin\left(\frac{2\pi x}{\lambda}\right) = A \sin\left(5 \times 10^{10} x\right)$ so $\left(\frac{2\pi}{\lambda}\right) = 5 \times 10^{10}\text{ m}^{-1}$,

$\lambda = \frac{2\pi}{5 \times 10^{10}} = 1.26 \times 10^{-10}\text{ m}$.

(b) $p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}\text{ Js}}{1.26 \times 10^{-10}\text{ m}} = 5.26 \times 10^{-24}\text{ kg m/s}$

(c) $K = \frac{p^2}{2m} \quad m = 9.11 \times 10^{-31}\text{ kg}$

$$K = \frac{\left(5.26 \times 10^{-24}\text{ kg m/s}\right)^2}{\left(2 \times 9.11 \times 10^{-31}\text{ kg}\right)} = 1.52 \times 10^{-17}\text{ J}$$

$$K = \frac{1.52 \times 10^{-17}\text{ J}}{1.6 \times 10^{-19}\text{ J/eV}} = 95\text{ eV}$$

6-6 $$\psi(x) = A \cos kx + B \sin kx$$

$$\frac{\partial \psi}{\partial x} = -kA \sin kx + kB \cos kx$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 A \cos kx - k^2 B \sin kx$$

$$\left(\frac{-2m}{\hbar^2}\right)(E - U)\psi = \left(\frac{-2mE}{\hbar^2}\right)(A \cos kx + B \sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{-2m}{\hbar^2}\right)(E - U)\psi$ or

$$-k^2\left(A\cos kx + B\sin kx\right) = \left(\frac{-2mE}{\hbar^2}\right)\left(A\cos kx + B\sin kx\right).$$

Therefore $E = \dfrac{\hbar^2 k^2}{2m}$.

6-9 $E_n = \dfrac{n^2 h^2}{8mL^2}$, so $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$

$$\Delta E = (3)\frac{\left(1240\ \text{eV nm}/c\right)^2}{8\left(938.28\times 10^6\ \text{eV}/c^2\right)\left(10^{-5}\ \text{nm}\right)^2} = 6.14\ \text{MeV}$$

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\ \text{eV nm}}{6.14\times 10^6\ \text{eV}} = 2.02\times 10^{-4}\ \text{nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-10 $E_n = \dfrac{n^2 h^2}{8mL^2}$

$$\frac{h^2}{8mL^2} = \frac{\left(6.63\times 10^{-34}\ \text{Js}\right)^2}{8\left(9.11\times 10^{-31}\ \text{kg}\right)\left(10^{-10}\ \text{m}\right)^2} = 6.03\times 10^{-18}\ \text{J} = 37.7\ \text{eV}$$

(a) $E_1 = 37.7\ \text{eV}$

$E_2 = 37.7\times 2^2 = 151\ \text{eV}$

$E_3 = 37.7\times 3^2 = 339\ \text{eV}$

$E_4 = 37.7\times 4^2 = 603\ \text{eV}$

(b) $hf = \dfrac{hc}{\lambda} = E_{n_i} - E_{n_f}$

$$\lambda = \frac{hc}{E_{n_i} - E_{n_f}} = \frac{1\,240\ \text{eV}\cdot\text{nm}}{E_{n_i} - E_{n_f}}$$

For $n_i = 4$, $n_f = 1$, $E_{n_i} - E_{n_f} = 603\ \text{eV} - 37.7\ \text{eV} = 565\ \text{eV}$, $\lambda = 2.19\ \text{nm}$

$n_i = 4$, $n_f = 2$, $\lambda = 2.75\ \text{nm}$

$n_i = 4$, $n_f = 3$, $\lambda = 4.70\ \text{nm}$

$n_i = 3$, $n_f = 1$, $\lambda = 4.12\ \text{nm}$

$n_i = 3$, $n_f = 2$, $\lambda = 6.59\ \text{nm}$

$n_i = 2$, $n_f = 1$, $\lambda = 10.9\ \text{nm}$

6-12 $\Delta E = \dfrac{hc}{\lambda} = \left(\dfrac{h^2}{8mL^2}\right)\left[2^2 - 1^2\right]$ and $L = \left[\dfrac{(3/8)h\lambda}{mc}\right]^{1/2} = 7.93\times 10^{-10}\ \text{m} = 7.93\ \text{Å}$.

6-13 (a) Proton in a box of width $L = 0.200\ \text{nm} = 2\times 10^{-10}\ \text{m}$

$$E_1 = \frac{h^2}{8m_pL^2} = \frac{\left(6.626\times 10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(1.67\times 10^{-27}\ \text{kg}\right)\left(2\times 10^{-10}\ \text{m}\right)^2} = 8.22\times 10^{-22}\ \text{J}$$

$$= \frac{8.22\times 10^{-22}\ \text{J}}{1.60\times 10^{-19}\ \text{J/eV}} = 5.13\times 10^{-3}\ \text{eV}$$

(b) Electron in the same box:

$$E_1 = \frac{h^2}{8m_e L^2} = \frac{\left(6.626 \times 10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(9.11 \times 10^{-31}\ \text{kg}\right)\left(2 \times 10^{-10}\ \text{m}\right)^2} = 1.506 \times 10^{-18}\ \text{J} = 9.40\ \text{eV}\,.$$

(c) The electron has a much higher energy because it is much less massive.

6-16 (a) $\psi(x) = A\sin\left(\frac{\pi x}{L}\right)$, $L = 3$ Å. Normalization requires

$$1 = \int_0^L |\psi|^2\,dx = \int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right)dx = \frac{LA^2}{2}$$

so $A = \left(\frac{2}{L}\right)^{1/2}$

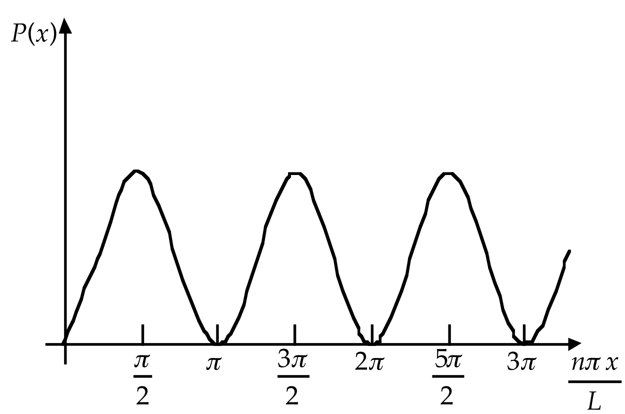
$$P = \int_0^{L/3} |\psi|^2\,dx = \left(\frac{2}{L}\right)\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^{\pi/3} \sin^2\phi\,d\phi = \frac{2}{\pi}\left[\frac{\pi}{6} - \frac{(3)^{1/2}}{8}\right] = 0.195\,5\,.$$

(b) $\psi = A\sin\left(\frac{100\pi x}{L}\right)$, $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \frac{2}{L}\int_0^{L/3} \sin^2\left(\frac{100\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L}{100\pi}\right)\int_0^{100\pi/3} \sin^2\phi\,d\phi = \frac{1}{50\pi}\left[\frac{100\pi}{6} - \frac{1}{4}\sin\left(\frac{200\pi}{3}\right)\right]$$
$$= \frac{1}{3} - \left[\frac{1}{200\pi}\right]\sin\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{400\pi} = 0.331\,9$$

(c) Yes: For large quantum numbers the probability approaches $\frac{1}{3}$.

6-18 Since the wavefunction for a particle in a one-dimension box of width $L$ is given by $\psi_n = A\sin\left(\frac{n\pi x}{L}\right)$ it follows that the probability density is $P(x) = |\psi_n|^2 = A^2\sin^2\left(\frac{n\pi x}{L}\right)$, which is sketched below:



From this sketch we see that $P(x)$ is a *maximum* when $\frac{n\pi x}{L} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = \pi\left(m + \frac{1}{2}\right)$ or when

$$x = \frac{L}{n}\left(m + \frac{1}{2}\right) \qquad m = 0,\ 1,\ 2,\ 3,\ \ldots,\ n.$$

Likewise, $P(x)$ is a *minimum* when $\frac{n\pi x}{L} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots = m\pi$ or when

$$x = \frac{Lm}{n} \qquad m = 0,\ 1,\ 2,\ 3,\ \ldots,\ n$$

6-29 (a) Normalization requires

$1 = \int_{-\infty}^{\infty} |\psi|^2\, dx = C^2 \int_0^{\infty} e^{-2x}\left(1 - e^{-x}\right)^2 dx = C^2 \int_0^{\infty}\left(e^{-2x} - 2e^{-3x} + e^{-4x}\right)dx$. The integrals are elementary and give $1 = C^2\left\{\frac{1}{2} - 2\left(\frac{1}{3}\right) + \frac{1}{4}\right\} = \frac{C^2}{12}$. The proper units for $C$ are those of $(\text{length})^{-1/2}$ thus, normalization requires $C = (12)^{1/2}\ \text{nm}^{-1/2}$.

(b) The most likely place for the electron is where the probability $|\psi|^2$ is largest. This is also where $\psi$ itself is largest, and is found by setting the derivative $\frac{d\psi}{dx}$ equal zero:

$$0 = \frac{d\psi}{dx} = C\left\{-e^{-x} + 2e^{-2x}\right\} = Ce^{-x}\left\{2e^{-x} - 1\right\}.$$

The RHS vanishes when $x = \infty$ (a minimum), and when $2e^{-x} = 1$, or $x = \ln 2\ \text{nm}$. Thus, the most likely position is at $x_p = \ln 2\ \text{nm} = 0.693\ \text{nm}$.

(c) The average position is calculated from

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2\, dx = C^2 \int_0^{\infty} xe^{-2x}\left(1 - e^{-x}\right)^2 dx = C^2 \int_0^{\infty} x\left(e^{-2x} - 2e^{-3x} + e^{-4x}\right)dx.$$

The integrals are readily evaluated with the help of the formula $\int_0^{\infty} xe^{-ax}\, dx = \frac{1}{a^2}$ to get $\langle x \rangle = C^2\left\{\frac{1}{4} - 2\left(\frac{1}{9}\right) + \frac{1}{16}\right\} = C^2\left\{\frac{13}{144}\right\}$. Substituting $C^2 = 12\ \text{nm}^{-1}$ gives

$$\langle x \rangle = \frac{13}{12}\ \text{nm} = 1.083\ \text{nm}.$$

We see that $\langle x \rangle$ is somewhat greater than the most probable position, since the probability density is skewed in such a way that values of $x$ larger than $x_p$ are weighted more heavily in the calculation of the average.

6-30 The possible particle positions within the box are weighted according to the probability density $|\psi|^2 = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$ The position is calculated as $\langle x\rangle = \int_0^L x|\psi|^2\, dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx$. Making the change of variable $\theta = \frac{nx}{L}$ (so that $d\theta = \frac{\pi\, dx}{L}$) gives $\langle x\rangle = \frac{2L}{\pi^2}\int_0^\pi \theta \sin^2 n\theta\, d\theta$. Using the trigonometric identity $2\sin^2\theta = 1 - \cos 2\theta$, we get $\langle x\rangle = \frac{L}{\pi^2}\left\{\int_0^\pi \theta\, d\theta - \int_0^\pi \theta\cos 2n\theta\, d\theta\right\}$. An integration by parts shows that the second integral vanishes, while the first integrates to $\frac{\pi^2}{2}$. Thus, $\langle x\rangle = \frac{L}{2}$, independent of $n$. For the computation of $\langle x^2\rangle$, there is an extra factor of $x$ in the integrand. After changing variables to $\theta = \frac{\pi x}{L}$ we get $\langle x^2\rangle = \frac{L^2}{\pi^3}\left\{\int_0^\pi \theta^2\, d\theta - \int_0^\pi \theta^2 \cos 2n\theta\, d\theta\right\}$. The first integral evaluates to $\frac{\pi^3}{3}$, the second may be integrated by parts twice to get

$$\int_0^\pi \theta^2 \cos 2n\theta\, d\theta = -\frac{1}{n}\int_0^\pi \theta \sin 2n\theta\, d\theta = \left(\frac{1}{2n^2}\right)\theta\cos 2n\theta\Big|_0^\pi = \frac{\pi}{2n^2}.$$

Then $\langle x^2\rangle = \frac{L^2}{\pi^3}\left\{\frac{\pi^3}{3} - \frac{\pi}{2n^2}\right\} = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$.

6-31 The symmetry of $|\psi(x)|^2$ about $x = 0$ can be exploited effectively in the calculation of average values. To find $\langle x\rangle$

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2\, dx$$

We notice that the integrand is antisymmetric about $x = 0$ due to the extra factor of $x$ (an odd function). Thus, the contribution from the two half-axes $x > 0$ and $x < 0$ cancel exactly, leaving $\langle x\rangle = 0$. For the calculation of $\langle x^2\rangle$, however, the integrand is symmetric and the half-axes contribute equally to the value of the integral, giving

$$\langle x\rangle = \int_0^\infty x^2|\psi|^2\, dx = 2C^2\int_0^\infty x^2 e^{-2x/x_0}\, dx.$$

Two integrations by parts show the value of the integral to be $2\left(\frac{x_0}{2}\right)^3$. Upon substituting for $C^2$, we get $\langle x^2\rangle = 2\left(\frac{1}{x_0}\right)(2)\left(\frac{x_0}{2}\right)^3 = \frac{x_0^2}{2}$ and $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left(\frac{x_0^2}{2}\right)^{1/2} = \frac{x_0}{\sqrt{2}}$. In calculating the probability for the interval $-\Delta x$ to $+\Delta x$ we appeal to symmetry once again to write

$$P = \int_{-\Delta x}^{+\Delta x} |\psi|^2 dx = 2C^2 \int_0^{\Delta x} e^{-2x/x_0} dx = -2C^2 \left(\frac{x_0}{2}\right) e^{-2x/x_0} \Big|_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

or about 75.7% independent of $x_0$.