6-20 The Schrödinger equation, after rearrangement, is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$. In the well interior, $U(x) = 0$ and solutions to this equation are $\sin kx$ and $\cos kx$, where $k^2 = \frac{2mE}{\hbar^2}$. The waves symmetric about the midpoint of the well ($x = 0$) are described by

$$\psi(x) = A\cos kx \qquad -L < x < +L$$

In the region outside the well, $U(x) = U$, and the independent solutions to the wave equation are $e^{\pm\alpha x}$ with $\alpha^2 = \left(\frac{2m}{\hbar^2}\right)(U - E)$.

(a) The growing exponentials must be discarded to keep the wave from diverging at infinity. Thus, the waves in the exterior region, which are symmetric about the midpoint of the well are given by

$$\psi(x) = Ce^{-\alpha|x|} \qquad x > L \text{ or } x < -L.$$

At $x = L$ continuity of ψ requires $A\cos kL = Ce^{-\alpha L}$. For the slope to be continuous here, we also must require $-Ak\sin kL = -Ce^{-\alpha L}$. Dividing the two equations gives the desired restriction on the allowed energies: $k\tan kL = \alpha$.

(b) The dependence on E (or k) is made more explicit by noting that $k^2 + \alpha^2 = \frac{2mU}{\hbar^2}$, which allows the energy condition to be written $k\tan kL = \left\{\frac{2mU}{\hbar^2} - k^2\right\}^{1/2}$. Multiplying by L, squaring the result, and using $\tan^2\theta + 1 = \sec^2\theta$ gives $(kL)^2\sec^2(kL) = \frac{2mUL^2}{\hbar^2}$ from which the desired form follows immediately, $k\sec(kL) = \frac{\sqrt{2mU}}{\hbar}$. The ground state is the symmetric waveform having the lowest energy. For electrons in a well of height $U = 5\text{ eV}$ and width $2L = 0.2\text{ nm}$, we calculate

$$\frac{2mUL^2}{\hbar^2} = \frac{(2)(511\times10^3\ \text{eV}/c^2)(5\text{ eV})(0.1\text{ nm})^2}{(197.3\ \text{eV}\cdot\text{nm}/c)^2} = 1.3127.$$

With this value, the equation for $\theta = kL$

$$\frac{\theta}{\cos\theta} = (1.3127)^{1/2} = 1.1457$$

can be solved numerically employing methods of varying sophistication. The simplest of these is trial and error, which gives $\theta = 0.799$ From this, we find $k = 7.99\text{ nm}^{-1}$, and an energy

$$E = \frac{\hbar^2k^2}{2m} = \frac{(197.3\ \text{eV}\cdot\text{nm}/c)^2(7.99\text{ nm}^{-1})^2}{2(511\times10^3\ \text{eV}/c^2)} = 2.432\text{ eV}.$$

6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$ with $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator. Differentiating $\psi(x) = Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx} = -2\alpha x\psi(x) + C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2} = -\frac{2\alpha x d\psi}{dx} - 2\alpha\psi(x) - (2\alpha x)Ce^{-\alpha x^2} = (2\alpha x)^2\psi(x) - 6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires $(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2}\{U(x) - E\} = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$.

Equating coefficients of like terms gives $2\alpha = \frac{m\omega}{\hbar}$ and $6\alpha = \frac{2mE}{\hbar^2}$. Thus, $\alpha = \frac{m\omega}{2\hbar}$ and $E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega$. The normalization integral is $1 = \int_{-\infty}^{\infty}|\psi(x)|^2 dx = 2C^2\int x^2 e^{-2\alpha x^2} dx$ where the second step follows from the symmetry of the integrand about $x = 0$. Identifying a with 2α in the integral of Problem 6-32 gives $1 = 2C^2\left(\frac{1}{8\alpha}\right)\left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-25 At its limits of vibration $x = \pm A$ the classical oscillator has all its energy in potential form: $E = \frac{1}{2}m\omega^2 A^2$ or $A = \left(\frac{2E}{m\omega^2}\right)^{1/2}$. If the energy is quantized as $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, then the corresponding amplitudes are $A_n = \left[\frac{(2n+1)\hbar}{m\omega}\right]^{1/2}$.

6-32 The probability density for this case is $|\psi_0(x)|^2 = C_0^2 e^{-ax^2}$ with $C_0 = \left(\frac{a}{\pi}\right)^{1/4}$ and $a = \frac{m\omega}{\hbar}$.

For the calculation of the average position $\langle x\rangle = \int_{-\infty}^{\infty} x|\psi_0(x)|^2 dx$ we note that the integrand is an odd function, so that the integral over the negative half-axis $x < 0$ exactly cancels that over the positive half-axis ($x > 0$), leaving $\langle x\rangle = 0$. For the calculation of $\langle x^2\rangle$, however, the integrand $x^2|\psi_0|^2$ is symmetric, and the two half-axes contribute equally, giving

$$\langle x^2\rangle = 2C_0^2\int_0^{\infty} x^2 e^{-ax^2} dx = 2C_0^2\left(\frac{1}{4a}\right)\left(\frac{\pi}{a}\right)^{1/2}.$$

Substituting for C_0 and a gives $\langle x^2\rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega}$ and $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.

6-33 (a) Since there is no preference for motion in the leftward sense vs. the rightward sense, a particle would spend equal time moving left as moving right, suggesting $\langle p_x \rangle = 0$.

(b) To find $\langle p_x^2 \rangle$ we express the average energy as the sum of its kinetic and potential energy contributions: $\langle E \rangle = \left\langle \frac{p_x^2}{2m} \right\rangle + \langle U \rangle = \frac{\langle p_x^2 \rangle}{2m} + \langle U \rangle$. But energy is sharp in the oscillator ground state, so that $\langle E \rangle = E_0 = \frac{1}{2}\hbar\omega$. Furthermore, remembering that $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator, and using $\langle x^2 \rangle = \frac{\hbar}{2m\omega}$ from Problem 6-32, gives $\langle U \rangle = \frac{1}{2}m\omega^2\langle x^2 \rangle = \frac{1}{4}\hbar\omega$. Then

$$\langle p_x^2 \rangle = 2m\left(E_0 - \langle U \rangle\right) = 2m\left(\frac{\hbar\omega}{4}\right) = \frac{m\hbar\omega}{2}.$$

(c) $$\Delta p_x = \left(\langle p_x^2 \rangle - \langle p_x \rangle^2\right)^{1/2} = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$$

6-34 From Problems 6-32 and 6-33, we have $\Delta x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$ and $\Delta p_x = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$. Thus,

$$\Delta x \Delta p_x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\frac{m\hbar\omega}{2}\right)^{1/2} = \frac{\hbar}{2}$$ for the oscillator ground state. This is the minimum uncertainty product permitted by the uncertainty principle, and is realized only for the ground state of the quantum oscillator.

6-35 Applying the momentum operator $[p_x] = \left(\frac{\hbar}{i}\right)\frac{d}{dx}$ to each of the candidate functions yields

(a) $$[p_x]\{A\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx)\}$$

(b) $$[p_x]\{A\sin(kx) - A\cos(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx) + A\sin(kx)\}$$

(c) $$[p_x]\{A\cos(kx) + iA\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{-A\sin(kx) + iA\cos(kx)\}$$

(d) $$[p_x]\left\{e^{ik(x-a)}\right\} = \left(\frac{\hbar}{i}\right)ik\left\{e^{ik(x-a)}\right\}$$

In case (c), the result is a multiple of the original function, since

$$-A\sin(kx) + iA\cos(kx) = i\{A\cos(kx) + iA\sin(kx)\}.$$

The multiple is $\left(\frac{\hbar}{i}\right)(ik) = \hbar k$ and is the eigenvalue. Likewise for (d), the operation $[p_x]$ returns the original function with the multiplier $\hbar k$. Thus, (c) and (d) are eigenfunctions of $[p_x]$ with eigenvalue $\hbar k$, whereas (a) and (b) are not eigenfunctions of this operator.

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{|(1/2)(1-i)|^2}{|(1/2)(1+i)|^2}$. But $|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2$, so that $R = 1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10$ MeV and $m = \frac{938.28\text{ MeV}}{c^2}$. Using $\hbar = 197.3\text{ MeV fm}/c\left(1\text{ fm} = 10^{-15}\text{ m}\right)$, we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3\text{ MeV fm}/c}{\left[(2)\left(938.28\text{ MeV}/c^2\right)(10\text{ MeV})\right]^{1/2}} = 1.44\text{ fm}.$$

7-2 (a) To the left of the step the particle is free with kinetic energy E and corresponding wavenumber $k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad x \le 0$$

To the right of the step the kinetic energy is reduced to $E - U$ and the wavenumber is now $k_2 = \left[\frac{2m(E-U)}{\hbar^2}\right]^{1/2}$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x \ge 0$$

with $D = 0$ for waves incident on the step from the left. At $x = 0$ both ψ and $\frac{d\psi}{dx}$ must be continuous: $\psi(0) = A + B = C$

$$\left.\frac{d\psi}{dx}\right|_0 = ik_1(A - B) = ik_2C.$$

(b) Eliminating C gives $A+B=\frac{k_1}{k_2}(A-B)$ or $A\left(\frac{k_1}{k_2}-1\right)=B\left(\frac{k_1}{k_2}+1\right)$. Thus,

$$R=\left|\frac{B}{A}\right|^2=\frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

$$T=1-R=\frac{4k_1k_2}{(k_1+k_2)^2}$$

(c) As $E\to U$, $k_2\to 0$, and $R\to 1$, $T\to 0$ (no transmission), in agreement with the result for any energy $E<U$. For $E\to\infty$, $k_1\to k_2$ and $R\to 0$, $T\to 1$ (perfect transmission) suggesting correctly that very energetic particles do not *see* the step and so are unaffected by it.

7-3 With $E=25\text{ MeV}$ and $U=20\text{ MeV}$, the ratio of wavenumber is $\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{25}{25-20}\right)^{1/2}=\sqrt{5}=2.236$. Then from Problem 7-2 $R=\frac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)^2}=0.146$ and $T=1-R=0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-4 The reflection coefficient for this case is given in Problem 7-2 as

$$R=\left|\frac{B}{A}\right|^2=\frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}.$$

The wavenumbers are those for electrons with kinetic energies $E=54.0\text{ eV}$ and $E-U=54.0\text{ eV}+10.0\text{ eV}=64.0\text{ eV}$:

$$\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{54\text{ eV}}{64\text{ eV}}\right)^{1/2}=0.918\,6.$$

Then, $R=\frac{(0.918\,6-1)^2}{(0.918\,6+1)^2}=1.80\times10^{-3}$ is the fraction of the incident beam that is reflected at the boundary.

7-5 (a) The transmission probability according to Equation 7.9 is $\frac{1}{T(E)}=1+\left[\frac{U^2}{4E(U-E)}\right]\sinh^2\alpha L$ with $\alpha=\frac{[2m(U-E)]^{1/2}}{\hbar}$. For $E<<U$, we find $(\alpha L)^2\approx\frac{2mUL^2}{\hbar^2}>>1$ by hypothesis. Thus, we may write $\sinh\alpha L\approx\frac{1}{2}e^{\alpha L}$. Also $U-E\approx U$, giving $\frac{1}{T(E)}\approx 1+\left(\frac{U}{16E}\right)e^{2\alpha L}\approx\left(\frac{U}{16E}\right)e^{2\alpha L}$ and a probability for transmission $P=T(E)=\left(\frac{16E}{U}\right)e^{-2\alpha L}$.

(b) Numerical Estimates: $(\hbar=1.055\times10^{-34}\text{ Js})$

1) For $m = 9.11 \times 10^{-31}$ kg, $U - E = 1.60 \times 10^{-21}$ J, $L = 10^{-10}$ m;

$\alpha = \frac{[2m(U-E)]^{1/2}}{\hbar} = 5.12 \times 10^{8}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.90$

2) For $m = 9.11 \times 10^{-31}$ kg, $U - E = 1.60 \times 10^{-19}$ J, $L = 10^{-10}$ m;

$\alpha = 5.12 \times 10^{9}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.36$

3) For $m = 6.7 \times 10^{-27}$ kg, $U - E = 1.60 \times 10^{-13}$ J, $L = 10^{-15}$ m;

$\alpha = 4.4 \times 10^{14}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.41$

4) For $m = 8$ kg, $U - E = 1$ J, $L = 0.02$ m; $\alpha = 3.8 \times 10^{34}\ \text{m}^{-1}$ and

$e^{-2\alpha L} = e^{-1.5 \times 10^{33}} \approx 0$

7-16 Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$, the first approximation to the decay length δ is

$$\delta \approx \frac{\hbar}{(2mU)^{1/2}} = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(30\ \text{MeV})\right]^{1/2}} = 0.415\,6\ \text{fm}.$$

This gives an effective width for the (infinite) well of $R + \delta = 9.415\,6$ fm, and a ground state energy $E_1 = \frac{\pi^2 (197.3\ \text{MeV fm}/c)^2}{2\left(3\,755.8\ \text{MeV}/c^2\right)(9.415\,6\ \text{fm})^2} = 0.577$ MeV. From this E we calculate $U - E = 29.42$ MeV and a new decay length

$$\delta = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(29.42\ \text{MeV})\right]^{1/2}} = 0.419\,7\ \text{fm}.$$

This, in turn, increases the effective well width to 9.419 7 fm and lowers the ground state energy to $E_1 = 0.576$ MeV. Since our estimate for E has changed by only 0.001 MeV, we may be content with this value. With a kinetic energy of E_1, the alpha particle in the ground state has speed $v_1 = \left(\frac{2E_1}{m}\right)^{1/2} = \left[\frac{2(0.576\ \text{MeV})}{\left(3\,755.8\ \text{MeV}/c^2\right)}\right]^{1/2} = 0.017\,5c$. In order to be ejected with a kinetic energy of 4.05 MeV, the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

7-17 The collision frequency f is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f = \frac{v}{2R}$, where v is the speed of the alpha. Now v is found from the kinetic energy which, inside the nucleus, is not the total energy E but the difference $E - U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R = 9$ fm, the Coulomb energy is

$$\frac{k(Ze)(2e)}{R} = 2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right) = 2(88)(27.2\ \text{eV})\left(\frac{5.29 \times 10^4\ \text{fm}}{9\ \text{fm}}\right) = 28.14\ \text{MeV}.$$

From this we conclude that $U = -1.86$ MeV to give a nuclear barrier of 30 MeV overall. Thus an alpha with $E = 4.05$ MeV has kinetic energy $4.05 + 1.86 = 5.91$ MeV inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$ this kinetic energy represents a speed

$$v = \left(\frac{2E_k}{m}\right)^{1/2} = \left[\frac{2(5.91)}{3\,755.8 \ \text{MeV}/c^2}\right]^{1/2} = 0.056c\,.$$

Thus, we find for the collision frequency $f = \dfrac{v}{2R} = \dfrac{0.056c}{2(9 \ \text{fm})} = 9.35 \times 10^{20} \ \text{Hz}$.