Physics 2D Final Exam Solutions Winter 2011

photon $\xleftarrow{}\ \xrightarrow{}$ ⊖ electron $\xrightarrow[K_e]{}$, $v = 0.6c$

$$K_e = \frac{1}{2} m_e v^2 = \frac{1}{2} m_e c^2 \left(\frac{v}{c}\right)^2 = \frac{1}{2} \times 0.511 \times 10^6 \, eV \times (0.6)^2$$

$$\Rightarrow \boxed{K_e = 91{,}980 \, eV \text{ classically}}$$

(b) In relativity, $K_e = (\gamma - 1) m_e c^2$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1.25 \Rightarrow K_e = 0.25 \, m_e c^2 \Rightarrow$$

$$\boxed{K_e = 127{,}750 \, eV}$$

(c) When the speed approaches c, the classical and relativistic answers differ, the relativistic one is always better.

(d) $$p_e = m_e v \gamma = m_e c^2 \frac{v}{c} \gamma = 0.511 \times 10^6 \times 0.6 \times 1.25 \, \frac{eV}{c}$$

$$\Rightarrow \boxed{p_e = 383{,}250 \, eV/c}$$

(e) $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) = \frac{2h}{m_e c}$ since $\theta = 180°$.

Energy conservation $\Rightarrow$ $\frac{hc}{\lambda} - \frac{hc}{\lambda'} = K_e = \frac{hc}{\lambda\lambda'}(\lambda' - \lambda) \Rightarrow$

$$\Rightarrow K_e = \frac{hc}{\lambda\lambda'} \cdot \frac{2h}{m_e c} \Rightarrow \lambda\lambda' = \frac{2(hc)^2}{m_e c^2 K_e} = \boxed{4.71 \times 10^{-3} \, \mathring{A}^2}$$

(f) we have: $\lambda' - \lambda = 4.86 \times 10^{-2} \, \mathring{A}$ and $\lambda\lambda' = 4.71 \times 10^{-3} \, \mathring{A}^2$

Solution: $\lambda' - \lambda = a$, $\lambda\lambda' = b \Rightarrow \lambda' = b/\lambda \Rightarrow \frac{b}{\lambda} - \lambda = a \Rightarrow b - \lambda^2 = a\lambda$

$$\Rightarrow \lambda^2 + a\lambda - b = 0 \Rightarrow \lambda = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + b} \Rightarrow \boxed{\lambda = 0.0485 \mathring{A}}$$

$$\Rightarrow \boxed{\lambda' = 0.0971 \mathring{A}}$$ Check: $\frac{hc}{\lambda} - \frac{hc}{\lambda'} = 127{,}967 \sim K_e$ ←

Problem 2

$$E_n = \frac{\hbar^2\pi^2}{2m_e L^2} n^2 = \frac{37.6\,eV\,Å^2}{25\,Å^2} n^2 = 1.504\,eV \cdot n^2$$

$\Rightarrow E_1 = 1.504\,eV, \quad E_2 = 6.016\,eV$

(b) For $n=3$, $E_3 = 9 \times 1.504\,eV = 13.54\,eV > 8\,eV$.

Since the energy is higher than the left barrier, electron will escape from the well.

(c) In the transition from $n=1$ to $n=2$,

$$E_2 - E_1 = 4.512\,eV = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda = \frac{12{,}400}{4.512}\,Å = 2748\,Å}$$

(d) The tunneling probability is given by $T = e^{-2\alpha \cdot \Delta x}$

For $n=1$ and the left barrier:

$$2\alpha\,\Delta x = 2\sqrt{\frac{2m_e}{\hbar^2}(U-E)}\,\Delta x = 2\sqrt{\frac{8-1.504}{3.81}} \cdot 2 = 5.22$$

$\Rightarrow T_{left} = e^{-5.22} = 5.4 \times 10^{-3}$

Right barrier: $2\alpha\,\Delta x = 2\sqrt{\frac{16-1.504}{3.81}} \times 1 = 3.90$

$\Rightarrow T_{right} = e^{-3.90} = 2.02 \times 10^{-2} \Rightarrow P_A/P_B = 0.27$, more likely to be at B.

For $n=2$, left barrier:

$$2\alpha\,\Delta x = 2\sqrt{\frac{8-6.016}{3.81}} \cdot 2 = 2.886 \Rightarrow T_l = 0.0558$$

Right: $2\alpha\,\Delta x = 2\sqrt{\frac{16-6.016}{3.81}} \times 1 = 3.24 \Rightarrow T_r = 0.039$

$\Rightarrow P_A/P_B = 1.43$, more likely to be at A.

(Prob. 2 cont.)

(f) In the forbidden regions, $\Psi \sim e^{-x/\delta}$, with $\delta = \frac{1}{\alpha}$

We can approximate

$$E_1 = \frac{\hbar^2 \pi^2}{2 m_e L_{eff}^2}$$

$$L_{eff} = L + \delta_{left} + \delta_{right} = L + \frac{1}{\alpha_{left}} + \frac{1}{\alpha_{right}}$$

$$\alpha_{left} = \sqrt{\frac{8 - 1.504}{3.81}}\ \mathring{A}^{-1} = 1.306\ \mathring{A}^{-1} \Rightarrow \delta_{left} = 0.766\ \mathring{A}$$

$$\alpha_{right} = \sqrt{\frac{16 - 1.504}{3.81}}\ \mathring{A}^{-1} = 1.95\ \mathring{A}^{-1} \Rightarrow \delta_{right} = 0.513\ \mathring{A}$$

$$\Rightarrow L_{eff} = 5\ \mathring{A} + 0.766\ \mathring{A} + 0.513\ \mathring{A} = 6.28\ \mathring{A}$$

$$E_1 = E_1(\infty \text{ well}) \cdot \frac{L^2}{L_{eff}^2} = 1.504\,eV \cdot \left(\frac{5\ \mathring{A}}{6.28\ \mathring{A}}\right)^2$$

$$\Rightarrow \boxed{E_1 = 0.95\,eV}$$

Problem 3

(a) $\langle x^2 \rangle = 2\,\mathring{A}^2$; $\langle x \rangle = 0$ by symmetry $\Rightarrow$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x \rangle^2} = \sqrt{2}\,\mathring{A} = 1.414\,\mathring{A}$$

(b) $\Delta x \Delta p = \hbar/2 \Rightarrow \Delta p = \dfrac{\hbar}{2\Delta x} = \dfrac{\hbar c}{2\Delta x c} = \dfrac{1973}{2\sqrt{2}} \dfrac{eV\mathring{A}}{\mathring{A} c} \Rightarrow$

$$\boxed{\Delta p = 697.6\, eV/c}$$

(c) $\langle p^2 \rangle = (\Delta p)^2$ since $\langle p \rangle = 0$ and $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

$$\langle K \rangle = \frac{\langle p^2 \rangle}{2m_e} = \frac{\hbar^2}{8m_e(\Delta x)^2} = \frac{3.81\, eV\mathring{A}^2}{8\,\mathring{A}^2} = 0.476\, eV$$

(d) Ground state energy is $E_0 = \dfrac{\hbar\omega}{2} = \langle K \rangle + \langle U \rangle$

Since $\langle K \rangle = \langle U \rangle \Rightarrow \langle K \rangle = \dfrac{\hbar\omega}{4} \Rightarrow \hbar\omega = 4\langle K \rangle$

$$\Rightarrow \boxed{\hbar\omega = 1.905\, eV}$$

(e) $\frac{1}{2} m_e \omega^2 A^2 = E_0 = \dfrac{\hbar\omega}{2}$; given that $\langle U \rangle = \frac{1}{2} m_e \omega^2 \langle x^2 \rangle = \dfrac{\hbar\omega}{4}$

$$\Rightarrow A^2 = 2\langle x^2 \rangle = 2 \times 2\,\mathring{A}^2 \Rightarrow$$

classical amplitude is $\boxed{A = 2\,\mathring{A}}$

Alternative solution / consistency check:

$$\frac{1}{\cancel{2}} m_e \omega^{\cancel{2}} A^2 = \frac{\hbar \cancel{\omega}}{\cancel{2}} \Rightarrow A^2 = \frac{\hbar}{m_e \omega} = \frac{\hbar^2}{m_e \hbar\omega} = \frac{2 \times 3.81}{1.905}\,\mathring{A}^2$$

$$\Rightarrow A^2 = 4\,\mathring{A}^2 \Rightarrow \boxed{A = 2\,\mathring{A}}$$

Problem 4

$\psi(r,\theta,\phi) = C r^3 e^{-r/a_0} \sin^2\theta \, e^{-2i\phi}$, $Z=3$

(a) General form has e^{-Zr/na_0} ; since $Z=3 \Rightarrow$

$\Rightarrow \boxed{n=3}$; clearly $\boxed{m_\ell=-2}$ since general form is $e^{im_\ell\phi}$.

Since $|m_\ell| \le \ell$ and $\ell \le n-1 \Rightarrow \boxed{\ell=2}$

Energy is $E_n = -\frac{ke^2}{2a_0}\frac{Z^2}{n^2} \Rightarrow \boxed{E_3=-13.6\,eV \text{ for } Z=3}$

(b) $E_3 = -E_0 = -13.6\,eV$

$E_2 = -E_0 \cdot \frac{9}{4} = -30.6\,eV$

$E_1 = -E_0 \cdot 9 = -122.4\,eV$

$\Rightarrow E_3 - E_2 = 17\,eV = hc/\lambda \Rightarrow \boxed{\lambda = 729.4\,\mathring{A}}$

$E_3 - E_1 = 108.8\,eV = hc/\lambda \Rightarrow \boxed{\lambda = 114.0\,\mathring{A}}$

$E_2 - E_1 = 91.8\,eV = hc/\lambda \Rightarrow \boxed{\lambda = 135.1\,\mathring{A}}$

(c) $\frac{P(\theta=\pi/2)}{P(\theta=\pi/4)} = \frac{|\psi(\theta=\pi/2)|^2}{|\psi(\theta=\pi/4)|^2} = \frac{\sin^4\pi/2}{\sin^4\pi/4} = \frac{1}{(1/\sqrt{2})^4} = \boxed{4}$

(d) $U = -\vec{\mu}\cdot\vec{B} = +\frac{e}{2m_e}L_z B = \frac{e\hbar}{2m_e} m_\ell B = \mu_B m_\ell B$; $B=5T$

For $m_\ell=-2$, energy decreases by $-2\mu_B B = \boxed{-5.79\times10^{-4}\,eV}$

(e) With spin, $U = \frac{e\hbar}{2m_e}(m_\ell + 2m_s)B = \mu_B(m_\ell+2m_s)B$

For $m_s=+\frac{1}{2}$, $U = \mu_B(-2+1)B = -\mu_B B = \boxed{-2.895\times10^{-4}\,eV}$

For $m_s=-1/2$, $U = \mu_B(-2-1)B = \boxed{-8.685\times10^{-4}\,eV}$

Problem 5

$$E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m_e}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}\right) = \frac{\hbar^2\pi^2}{2m_e L_1^2}\left(n_1^2 + \frac{L_1^2}{L_2^2}n_2^2\right) \Rightarrow$$

$$E_{n_1,n_2} = 37.6\,\text{eV}\left(n_1^2 + 2n_2^2\right) \equiv E_0\left(n_1^2 + 2n_2^2\right)$$

$(1,1)$: $3E_0$; $E_{11}/E_{11} = 1$

$(2,1)$: $6E_0$; $E_{21}/E_{11} = 2$

$(1,2)$: $9E_0$; $E_{1,2}/E_{1,1} = 3$

$(3,1)$: $11E_0$; $E_{3,1}/E_{1,1} = 11/3 = 3.67$

$(2,2)$: $12E_0$; $E_{2,2}/E_{1,1} = 4$

(a) There are two electrons per energy level maximum.

$\Rightarrow$ for 9 electrons, there are two in states $(1,1)$, $(2,1)$, $(1,2)$, $(3,1)$, and one electron in state $(2,2)$.

Total energy:

$$E = 2E_{11} + 2E_{21} + 2E_{12} + 2E_{31} + E_{22} =$$

$$= 6E_0 + 12E_0 + 18E_0 + 22E_0 + 12E_0 =$$

$$= 70E_0 = \boxed{2{,}632\,\text{eV}}$$

(b) $$\psi_{n_1n_2}(x,y) = C\sin\frac{n_1\pi x}{L_1}\sin\frac{n_2\pi y}{L_2}.$$

For $\psi = 0$ at $x = L_1/2$, need $n_1 = 2, 4, 6, \ldots$ So two lowest states are $(2,1)$ and $(2,2)$: $\boxed{E_{21} = 225.6\,\text{eV}}$, $\boxed{E_{22} = 451.2\,\text{eV}}$

$$\boxed{\begin{aligned}\psi_{21}(x,y) &= C\sin\frac{2\pi x}{L_1}\sin\frac{\pi y}{L_2}\\ \psi_{22}(x,y) &= C\sin\frac{2\pi x}{L_1}\sin\frac{2\pi y}{L_2}\end{aligned}}$$

Problem 6

(a) $P(r) = C^2 r^4 e^{-r/a_0}$; most probable r satisfies $P'(r) = 0$

$$P'(r) = \left(4r^3 - \frac{r^4}{a_0}\right) C^2 e^{-r/a_0} = 0 \implies \boxed{r = 4a_0}$$

(b) In the Bohr model, $\boxed{r = n^2 a_0}$, so for $n=2$, $r = 4a_0$, same as (a). The Bohr model gives the most probable radius for the states with $l = n-1$. That is the case here, with $l=1, n=2$.

(c) Using that $C^2 = \frac{1}{4!\,a_0^5}$ and $\int dr\, r^n e^{-\lambda r} = \frac{n!}{\lambda^{n+1}}$,

$$\langle r \rangle = \int_0^\infty dr\, r\, P(r) = C^2 \int_0^\infty dr\, r^5 e^{-r/a_0} = \frac{1}{4!\,a_0^5} \cdot 5!\, a_0^6 = \boxed{5a_0}$$

$$\langle r^2 \rangle = C^2 \int_0^\infty dr\, r^6 e^{-r/a_0} = \frac{1}{4!\,a_0^5}\, 6!\, a_0^7 = \boxed{30 a_0^2}$$

$$\implies \Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2} = \sqrt{30a_0^2 - 25a_0^2} = \boxed{\sqrt{5}\,a_0 = 2.24\,a_0}$$

(d) $p = m_e v$; using $L = m_e v r = n\hbar \Rightarrow pr = n\hbar \Rightarrow p = n\hbar/r$

$$\implies p = n\hbar/n^2 a_0 = \hbar/n a_0 \implies \boxed{p = \frac{\hbar}{2a_0}}$$

(e) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$; $\langle p \rangle = 0$; using $\langle p^2 \rangle = \frac{\hbar^2}{4a_0^2}$ (Bohr atom value)

$$\implies \boxed{\Delta p = \frac{\hbar}{2a_0}} \implies \Delta r\, \Delta p = \frac{\hbar}{2a_0} \cdot 2.24\, a_0 \implies$$

$$\implies \boxed{\Delta r\, \Delta p = 1.12\,\hbar}$$

It is in agreement with the uncertainty principle $\Delta x \Delta p \sim \hbar$ or $\Delta x \Delta p \geq \hbar/2$.