PHYS 4D
Solution to HW 8

February 25, 2011

Problem Giancoli 36-5 (II) What is the speed of a pion if its average lifetime is measured to be $4.40 \times 10^{-8}s$? At rest, its average lifetime is $2.60 \times 10^{-8}s$.

Solution: The speed is determined from the time dilation relation, Eq. 36-1a.

$$\Delta t_0 = \Delta t\sqrt{1 - v^2/c^2} \Rightarrow v = c\sqrt{1 - \left(\frac{\Delta t_0}{\Delta t}\right)^2} = c\sqrt{1 - \left(\frac{2.60 \times 10^{-8}s}{4.40 \times 10^{-8}s}\right)^2} = 0.807c = 2.42 \times 10^8 m/s.$$

Problem Giancoli 36-7 (II) Suppose you decide to travel to a star 65 light-years away at a speed that tells you the distance is only 25 light-years. How many years would it take you to make the trip?

Solution: The speed is determined from the length contraction relation, Eq. 36-3a.

$$l = l_0\sqrt{1 - v^2/c^2} \Rightarrow v = c\sqrt{1 - \left(\frac{l}{l_0}\right)^2} \Rightarrow t = \frac{l}{v} = \frac{l}{c\sqrt{1 - \left(\frac{l}{l_0}\right)^2}} = \frac{25ly}{c\sqrt{1 - \left(\frac{25ly}{65ly}\right)^2}} = \frac{(25y)\,c}{c\,(0.923)} = 27y.$$

Problem Giancoli 36-12 (II) A certain star is 18.6 light-years away. How long would it take a spacecraft traveling $0.950c$ to reach that star from Earth, as measured by observers: (a) on Earth, (b) on the spacecraft? (c) What is the distance traveled according to observers on the spacecraft? (d) What will the spacecraft occupants compute their speed to be from the results of (b) and (c)?

Solution: (a) $l_0 = 18.6ly$.

$$t_{Earth} = \frac{l_0}{v} = \frac{18.6ly}{0.950c} = 19.58yr.$$

(b) The time as observed on the spacecraft is shorter. Use Eq. 36-1a.

$$\Delta t_0 = \Delta t\sqrt{1 - v^2/c^2} = 19.58yr\sqrt{1 - (0.95)^2} = 6.11yr.$$

(c) To the spacecraft observer, the distance to the star is contracted. Use Eq. 36-3a.

$$l = l_0\sqrt{1 - v^2/c^2} = 18.6ly\sqrt{1 - (0.95)^2} = 5.81ly.$$

(d) To the spacecraft observer, the speed of the spacecraft is their observed distance divided by their observed time.

$$v = \frac{l}{\Delta t_0} = \frac{5.81ly}{6.114yr} = 0.95c.$$

Problem Giancoli 36-17 (II) When at rest, a spaceship has the form of an isosceles triangle whose two equal sides have length $2l$ and whose base has length l. If this ship flies past an observer with a relative velocity of $v = 0.95c$ directed along its base, what are the lengths of the ship's three sides according to the observer?

Solution: The vertical dimensions of the ship will not change, but the horizontal dimensions will be contracted according to Eq. 36-3a. The base will be contracted as follows.

$$l_{base} = l\sqrt{1 - v^2/c^2} = 0.31l.$$

When at rest, the angle of the sides with respect to the base is given by $\theta = \cos^{-1}\frac{0.50l}{2.0l} = 75.52°$. The vertical component of $l_{vert} = 2l\sin\theta = 2l\sin 75.52° = 1.936l$ is unchanged. The horizontal component, which is $2l\cos\theta = 2l(\frac{1}{4}) = 0.50l$ at rest, will be contracted in the same way as the base.

$$l_{horizontal} = 0.50l\sqrt{1 - v^2/c^2} = 0.50l\sqrt{1 - (0.95)^2} = 0.156l.$$

Use the Pythagorean theorem to find the length of the leg.

$$l_{leg} = \sqrt{l_{horizontal}^2 + l_{vert}^2} = \sqrt{(0.156l)^2 + (1.936l)^2} = 1.94l.$$

Problem Giancoli 36-21 (I) Repeat Problem 20 using the Lorentz transformation and a relative speed $v = 1.80 \times 10^8 m/s$, but choose the time t to be (a) $3.5\mu s$ and (b) $10.0\mu s$.

Solution: (a) The person's coordinates in S are found using Eq. 36-6, with $x' = 25m$, $y' = 20m$, $z' = 0$, and $t' = 3.5\mu s$. We set $v = 1.8 \times 10^8 m/s$.

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = 820m, \\ y &= y', z = z'. \end{aligned}$$

(b) We repeat part (a) using the time $t' = 10.0\mu s$.

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = 2280m, \\ y &= y', z = z'. \end{aligned}$$

Problem Giancoli 36-23 (II) Two spaceships leave Earth in opposite directions, each with a speed of $0.60c$ with respect to Earth. (a) What is the velocity of spaceship 1 relative to spaceship 2? (b) What is the velocity of spaceship 2 relative to spaceship 1?

Solution:

(a) We take the positive direction to be the direction of motion of spaceship 1. Consider spaceship 2 as reference frame S, and the Earth reference frame S'. The velocity of the Earth relative to spaceship 2 is $v = 0.60c$. The velocity of spaceship 1 relative to the Earth is $u'_x = 0.60c$. Solve for the velocity of spaceship 1 relative to spaceship 2, u_x, using Eq. 36-7a.

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{0.60c + 0.60c}{1 + 0.60 \times 0.60} = 0.88c.$$

(b) Now consider spaceship 1 as reference frame S. The velocity of the Earth relative to spaceship 1 is $v = -0.60c$. The velocity of spaceship 2 relative to the Earth is $u'_x = -0.60c$. Solve for the velocity of spaceship 2 relative to spaceship 1, u_x, using Eq. 36-7a.

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{-0.60c - 0.60c}{1 + (-0.60) \times (-0.60)} = -0.88c.$$

Problem Giancoli 36-25 (II) A spaceship leaves Earth traveling at $0.61c$. A second spaceship leaves the first at a speed of $0.87c$ with respect to the first. Calculate the speed of the second ship with respect to Earth if it is fired (a) in the same direction the first spaceship is already moving, (b) directly backward toward Earth.

Solution: (a) We take the positive direction in the direction of the first spaceship.

Reference frame S: the Earth,

Reference frame S': the first spaceship,

$v = 0.61c$.

The speed of the second spaceship relative to the first spaceship is $u'_x = 0.87c$. We use Eq. 36-7a to solve for the speed of the second spaceship relative to the Earth, u.

$$u_x = \frac{(u'_x + v)}{\left(1 + \frac{vu'_x}{c^2}\right)} = \frac{(0.87c + 0.61c)}{(1 + 0.87 \times 0.61)} = 0.97c.$$

(b) The only difference is now that $u'_x = -0.87c$.

$$u_x = \frac{(u'_x + v)}{\left(1 + \frac{vu'_x}{c^2}\right)} = \frac{(-0.87c + 0.61c)}{(1 + (-0.87) \times 0.61)} = -0.55c.$$

Speed: $0.55c$.

Problem Giancoli 36-29 (II) A stick of length l_0, at rest in reference frame S, makes an angle θ with the x axis. In reference frame S', which moves to the right with velocity $\vec{\mathbf{v}} = v\hat{\imath}$ with respect to S, determine (a) the length l of the stick, and (b) the angle θ' it makes with the x' axis.

Solution: (a) In frame S' the horizontal component of the stick length will be contracted, while the vertical component remains the same. We use the trigonometric relations to determine the x and y components of the length of the stick. Then using Eq. 36-3a we determine the contracted length of the x-component. Finally, we use the Pythagorean theorem to determine stick length in frame S'.

$$\begin{aligned}
l_x &= l_0 \cos\theta; l_y = l_0 \sin\theta = l'_y; l'_x = l_x\sqrt{1 - v^2/c^2} = l_0 \cos\theta\sqrt{1 - v^2/c^2}, \\
l &= \sqrt{l'^2_x + l'^2_y} = \sqrt{l_0^2 \cos^2\theta\,(1 - v^2/c^2) + l_0^2 \sin^2\theta} = l_0\sqrt{1 - (v\cos\theta/c)^2}.
\end{aligned}$$

Problem Giancoli 36-31 (III) Two lightbulbs, A and B, are placed at rest on the x axis at positions $x_A = 0$ and $x_A = +l$. In this reference frame, the bulbs are turned on simultaneously. Use the Lorentz transformations to find an expression for the time interval between when the bulbs are turned on as measured by an observer moving at velocity v in the $+x$ direction. According to this observer, which bulb is turned on first?

Solution:
S': moving with the observer,
S: light bulbs at rest,
v: velocity of S moving respect to frame S'.
We solve Eq. 36-6 for $t'(t, x, v)$.
S': $t_A = ?$, $t_B = ?$ that each bulb is turned on,
S: the bulbs are turned on simultaneously at $t_A = t_B = 0$.

$$\begin{aligned}
x &= \gamma(x' + vt') \Rightarrow x' = \frac{x}{\gamma} - vt' \\
t &= \gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma\left(t' + \frac{v}{c^2}\left(\frac{x}{\gamma} - vt'\right)\right) = \gamma t'\left(1 - \frac{v^2}{c^2}\right) + \frac{v}{c^2}x = \frac{t'}{\gamma} + \frac{vx}{c^2}. \\
&\Rightarrow t' = \gamma\left(t - \frac{vx}{c^2}\right). \\
&\Rightarrow t'_A = \gamma\left(t_A - \frac{vx_A}{c^2}\right) = 0, t'_B = \gamma\left(t_B - \frac{vx_B}{c^2}\right) = \gamma\left(0 - \frac{vl}{c^2}\right) = -\gamma\frac{vl}{c^2}. \\
\Delta t' &= t'_B - t'_A = -\gamma\frac{vl}{c^2}.
\end{aligned}$$

According to the observer, bulb B turned on first.

Problem Giancoli 36-37 (II) An unstable particle is at rest and suddenly decays into two fragments. No external forces act on the particle or its fragments. One of the fragments has a speed of $0.60c$ and a mass of $6.68 \times 10^{-27}kg$, while the other has a mass of $1.67 \times 10^{-27}kg$. What is the speed of the less massive fragment?

Solution: The two momenta, as measured in the frame in which the particle was initially at rest, will be equal to each other in magnitude. The lighter (heavier) particle is designated with a subscript "1" ("2").

$$\begin{aligned}
p_1 &= p_2 \Rightarrow \frac{m_1 v_1}{\sqrt{1 - v_1^2/c^2}} = \frac{m_2 v_2}{\sqrt{1 - v_2^2/c^2}}, \\
\frac{v_1^2}{1 - v_1^2/c^2} &= \left(\frac{m_2}{m_1}\right)^2 \frac{v_2^2}{1 - v_2^2/c^2} = \left(\frac{6.68 \times 10^{-27}kg}{1.67 \times 10^{-27}kg}\right)^2 \frac{(0.60c)^2}{1 - (0.60)^2} = 9.0c^2. \\
&\Rightarrow v_1 = \sqrt{0.90}c = 0.95c.
\end{aligned}$$

Problem Giancoli 36-41 (I) The total annual energy consumption in the United States is about $8 \times 10^{19} J$. How much mass would have to be converted to energy to fuel this need?

Solution: We find the mass conversion from Eq. 36-12.

$$m = \frac{E}{c^2} = \frac{\left(8 \times 10^{19} J\right)}{(3 \times 10^8 m/s)^2} = 900kg.$$

Problem Giancoli 36-44 (II) (a) How much work is required to accelerate a proton from rest up to a speed of $0.998c$? (b) What would be the momentum of this proton?

Solution: (a) The work is the change in kinetic energy. Use Eq. 36-10b. The initial kinetic energy is 0.

$$W = \Delta K = K_{final} = (\gamma - 1) mc^2 = \left(\frac{1}{\sqrt{1 - 0.998^2}} - 1\right)(938.3MeV) = 1.39 \times 10^4 MeV.$$

(b) The momentum of the proton is given by Eq. 36-8.

$$p = \gamma mv = \frac{1}{\sqrt{1 - 0.998^2}} \left(938.3MeV/c^2\right)(0.998c) = 1.48 \times 10^4 MeV.$$

Problem Giancoli 36-46 (II) To accelerate a particle of mass m from rest to speed $0.90c$ requires work W_1. To accelerate the particle from speed $0.90c$ to $0.99c$, requires work W_2. Determine the ratio W_2/W_1.

Solution: The work is the change in kinetic energy. Use Eq. 36-10b. The initial kinetic energy is 0.

$$\begin{aligned} W_1 &= (\gamma_{0.90} - 1) mc^2; W_2 = K_{0.99c} - K_{0.90c} = (\gamma_{0.99} - 1) mc^2 - (\gamma_{0.90} - 1) mc^2. \\ W_2/W_1 &= 3.7. \end{aligned}$$

Problem Giancoli 36-55 (II) Calculate the speed of a proton ($m = 1.67 \times 10^{-27} kg$) whose kinetic energy is exactly half (a) its total energy, (b) its rest energy.

Solution: (a) Since the kinetic energy is half the total energy, and the total energy is the kinetic energy plus the rest energy, the kinetic energy must be equal to the rest energy. We also use Eq. 36-10.

$$\begin{aligned} K &= \frac{1}{2}E = \frac{1}{2}\left(K + mc^2\right) \Rightarrow K = mc^2. \\ K &= (\gamma - 1) mc^2 = mc^2 \Rightarrow \gamma = 2 = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow v = \sqrt{\frac{3}{4}}c = 0.866c. \end{aligned}$$

(b) In this case, the kinetic energy is half the rest energy.

$$K = (\gamma - 1) mc^2 = \frac{1}{2}mc^2 \Rightarrow \gamma = \frac{3}{2} = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow v = \sqrt{\frac{5}{9}}c = 0.745c.$$

Problem Giancoli 36-60 (II) Make a graph of the kinetic energy versus momentum for (a) a particle of nonzero mass, and (b) a particle with zero mass.

Solution: (a) For a particle of non-zero mass, we derive the following relationship between kinetic energy and momentum.

$$\begin{aligned} E &= K + mc^2; (pc)^2 = E^2 - \left(mc^2\right)^2 = \left(K + mc^2\right)^2 - \left(mc^2\right)^2 = K^2 - 2K\left(mc^2\right), \\ K^2 + 2K(mc) - (pc)^2 &= 0 \Rightarrow K = \frac{-2mc^2 \pm \sqrt{4\left(mc^2\right)^2 + 4(pc)^2}}{2}. \end{aligned}$$

For the kinetic energy to be positive, we take the positive root.

$$K = \frac{-2mc^2 + \sqrt{4\left(mc^2\right)^2 + 4(pc)^2}}{2} = -mc^2 + \sqrt{\left(mc^2\right)^2 + (pc)^2}.$$

If the momentum is large, we have the following relationship.

$$K = -mc^2 + \sqrt{\left(mc^2\right)^2 + \left(pc\right)^2} \approx pc - mc^2.$$

Thus, there should be a linear relationship between kinetic energy and momentum for large values of momentum.

If the momentum is small, we use the binomial expansion to derive the classical relationship.

$$K = -mc^2 + mc^2\sqrt{1 + \left(\frac{pc}{mc^2}\right)^2} \approx -mc^2 + mc^2\left(1 + \frac{1}{2}\left(\frac{pc}{mc^2}\right)^2\right) = \frac{p^2}{2m}.$$

Thus we expect a quadratic relationship for small values of momentum. The adjacent graph verifies these approximations.

(b) For a particle of zero mass, the relationship is simply $K = pc$. ...