Solutions to Assignment 5, UCSD Physics 130b

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1. WKB: Harmonic Oscillator

   Let us consider the harmonic oscillator, with potential $V = \frac{1}{2}m\omega^2 x^2$. For a particle with total energy, E, we find that there are two classical turning points, $x_\pm = \pm x_0$, where $x_0 \equiv \sqrt{\frac{2E}{m\omega^2}}$. We see that the momentum can be writen as $p(x) = m\omega\sqrt{x_0^2 - x^2}$. Using the connection formulas for both the right and left turning point, we find the quantization condition,

   $$\int_{x_-}^{x_+} dx\, p(x) = \hbar\pi\left(n - \tfrac{1}{2}\right) \tag{1}$$

   Using our formula for momentum, and making the substitution $x = \cos\theta$,

   $$\tfrac{E}{\omega}\int_0^\pi d\theta\,(1 - \cos 2\theta) = \tfrac{E}{\omega}\pi = \hbar\pi\left(n - \tfrac{1}{2}\right) \tag{2}$$

   It is clear that, $E_n = \hbar\omega\left(n - \frac{1}{2}\right)$. This is the same as the expected harmonic oscillator solution, with the caveat that here, the limitation of n is only that it is an integer, not that it must be positive.

2. Electron Tunneling

   The potential, a simple model for binding, is given by figure 1. Before time zero, the electron, with energy of 10 eV, is trapped in the well between 0 and $2a_0$. After time zero, the potential is modified, making it possible for the electron to tunnel through a barrier, with classical turning points at $2a_0$ and $2.5a_0$. The validity for use of WKB approximation in this problem can be considered by the condition,

   $$\frac{1}{k^2}\frac{dk}{dx} \ll 1 \tag{3}$$

   This condition is plotted in figure 2. We see that the WKB approximation is technically not valid for calculating a wavefunction in the region $2a_0$ to $3a_0$. However, since we are only interested in the tunneling probability, which only depends on the initial wavefunction inside the well and the outgoing wavefunction, connected via an evanescent wave, this method is still valid.

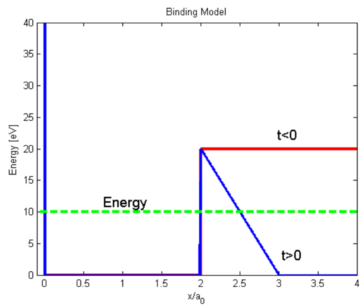

Figure 1: Potential barrier model for atom.

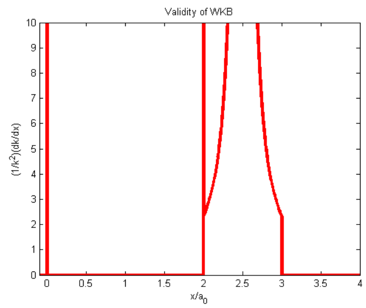

Figure 2: WKB is valid in regions where function is much less than 1

The transmission probability is given by $T \approx e^{-2\gamma}$, $\gamma = \frac{1}{\hbar}\int_{2a_0}^{2.5a_0} dx\, |p(x)|$, $|p(x)| = \sqrt{2m(V(x) - E)}$.

$$\gamma = \frac{\sqrt{20m}}{\hbar}\int_{2a_0}^{2.5a_0} dx\, \sqrt{(5 - 2\frac{x}{a_0})} = \frac{a_0\sqrt{20m}}{3\hbar}\left[(5 - 2\frac{x}{a_0})\right]^{3/2}\Bigg|_{2.5a_0}^{2a_0} = \frac{a_0\sqrt{20m}}{3\hbar} \tag{4}$$

If we use a_0, as the Bohr radius, $\gamma = 0.285$ and $T = 0.565$. The lifetime is given by

$$\tau = \frac{4a_0}{v}e^{2\gamma} = \frac{4a_0}{T}\sqrt{\frac{m}{2E}} = 2.00 \times 10^{-16}\text{sec.} \tag{5}$$

Here, the lifetime is extrememly short due to the large velocity compared to the bohr radius.