Solutions to Assignment 6, UCSD Physics 130b

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1. PDG

   See Table of Contents, particularly Clebsch-Gordan coefficients and Spherical harmonics on pg 299. The magnetic moment is given by $\mu \propto \frac{g}{m}$. Both the mass and the magnetic moment anomaly is known with better precision for the electron than the muon. Recall, that for products, the propagation of error can be written $(\frac{\sigma_f}{f})^2 = \sum_j (\frac{\sigma_{g_j}}{g_j})^2_j$, where $f = \prod_j g_j$. Thus, $\frac{\sigma_{\mu_e}}{\mu_e} = 2.2 \times 10^{-8}$, dominated by the error in mass. Likewise, $\frac{\sigma_{\mu_\mu}}{\mu_\mu} = 3.3 \times 10^{-8}$, again dominated by the error in mass. It is difficult to compare the uncertainty of the electron's magnetic and dipole moment, since they have different units. Although, the electric dipole moment is known to more significant digits in cgs units, the normalized magnetic moment is better known than the normalized dipole moment. All leptons are spin-$\frac{1}{2}$ particles.

2. Griffiths 2.11

   Consider the harmonic oscillator with solutions, $\psi_0 = \alpha e^{\xi^2/2}$, $\psi_1 = \sqrt{2}\alpha\xi e^{\xi^2/2}$. Using the Gaussian integrals, we see that the expectation value of position operators

$$\left\langle 0|\hat{x}^l|0\right\rangle = \begin{cases} \frac{(l)!}{2^l(l/2)!}\left(\frac{\hbar}{m\omega}\right)^{l/2} & l \text{ even} \\ 0 & l \text{ odd} \end{cases} \tag{1}$$

$$\left\langle 1|\hat{x}^l|1\right\rangle = \begin{cases} \frac{(l+2)!}{2^{l+1}(l/2+1)!}\left(\frac{\hbar}{m\omega}\right)^{l/2} & l \text{ even} \\ 0 & l \text{ odd} \end{cases} \tag{2}$$

   Using the momentum operator $\hat{p} = \frac{\hbar}{i}\sqrt{\pi}\alpha^2\frac{\partial}{\partial\xi}$ and the properties of Hermite Polynomials

$$\left\langle n|\hat{p}^l|n\right\rangle = (i\hbar\sqrt{\pi}\alpha^2)^l \sum_{k=0}^{l\leq n} \binom{l}{k} \left[\frac{2^k n!}{(n-k)!}\right]^{\frac{1}{2}} (-1)^k \left\langle n|H_{l-k}(\tfrac{\xi}{\sqrt{2}})|n-k\right\rangle \tag{3}$$

   Solving this explicitly for $n = 0$,

$$\left\langle 0|\hat{p}^l|0\right\rangle = \begin{cases} \frac{(l-1)!!}{2}(m\hbar\omega)^{\frac{l}{2}} & l \text{ even} \\ 0 & l \text{ odd} \end{cases} \tag{4}$$

Similarly for $n = 1$

$$\langle 1|\hat{p}^l|1\rangle = \begin{cases} \frac{3(l-1)!!}{2}(m\hbar\omega)^{\frac{l}{2}} & l \text{ even} \\ 0 & l \text{ odd} \end{cases} \tag{5}$$

Recall that the variance is $\sigma_q^2 =< q^2 > - < q >^2$. Therefore for $n = 0$,

$$\begin{aligned} \sigma_x &= \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} \\ \sigma_p &= \left(\frac{m\hbar\omega}{2}\right)^{\frac{1}{2}} \\ \sigma_x\sigma_p &= \frac{\hbar}{2} \geq \frac{\hbar}{2} \end{aligned} \tag{6}$$

Similarly, for n = 1,

$$\begin{aligned} \sigma_x &= \left(\frac{3\hbar}{2m\omega}\right)^{\frac{1}{2}} \\ \sigma_p &= \left(\frac{3m\hbar\omega}{2}\right)^{\frac{1}{2}} \\ \sigma_x\sigma_p &= \frac{3\hbar}{2} \geq \frac{\hbar}{2} \end{aligned} \tag{7}$$

We can obtain the kinetic energy,

$$< T >= \frac{<p^2>}{2m} = \begin{cases} \frac{1}{4}\hbar\omega & \text{n} = 0 \\ \frac{3}{4}\hbar\omega & \text{n} = 1 \end{cases} \tag{8}$$

and potential energy,

$$< V >= \frac{m\omega^2 <x^2>}{2} = \begin{cases} \frac{1}{4}\hbar\omega & \text{n} = 0 \\ \frac{3}{4}\hbar\omega & \text{n} = 1 \end{cases} \tag{9}$$

Thus, the total energy

$$< H >=< K > + < V >= \begin{cases} \frac{1}{2}\hbar\omega & = E_0 \text{ n} = 0 \\ \frac{3}{2}\hbar\omega & = E_1 \text{ n} = 1 \end{cases} \tag{10}$$

3. Griffiths 9.2

Let us solve the two state time evolution equations for a time-independent purturbing Hamiltonian, H'.

$$\begin{aligned} \dot{c}_a(t) &= -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b(t) \\ \dot{c}_b(t) &= -\frac{i}{\hbar}H'_{ba}e^{+i\omega_0 t}c_a(t) \end{aligned} \tag{11}$$

Let us turn these two first-order differential equation into a single second-order differential equation by differentiating one of them and substituting,

$$\ddot{c}_b - i\omega_0\dot{c}_b + \frac{1}{\hbar^2}|H'_{ab}|^2 c_b = 0 \tag{12}$$

We have used the fact that $H'^{*}_{ba} = H'_{ab}$. This linear homogeneous second-order differential equation with constant coefficients has the solution,

$$c_b(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t} \tag{13}$$

where $\lambda_\pm = i\frac{\omega_0 \pm \omega}{2}$, and $\omega \equiv \sqrt{\omega_0^2 + \frac{4}{\hbar^2}|H'_{ab}|^2}$. Since $c_b(0) = 0$ we see that $A = -B$ and thus we can rewrite this solution as,

$$c_b(t) = Ae^{i\frac{\omega_0}{2}t} \sin\left(\tfrac{\omega}{2}t\right) \tag{14}$$

We can find c_a from $\dot{c}_b$, where $\dot{c}_b = Ae^{i\frac{\omega_0}{2}t}\left[\frac{\omega}{2}\cos\left(\frac{\omega}{2}t\right) + i\frac{\omega_0}{2}\sin\left(\frac{\omega}{2}t\right)\right]$. Therefore,

$$c_a(t) = i\frac{\hbar\omega}{2H'_{ba}}Ae^{-i\frac{\omega_0}{2}t}\left[\cos\left(\tfrac{\omega}{2}t\right) + i\frac{\omega_0}{\omega}\sin\left(\tfrac{\omega}{2}t\right)\right] \tag{15}$$

We have the initial condition that $c_a(0) = 1$, thus $A = -i\frac{2H'_{ba}}{\hbar\omega}$. We see that

$$|c_a(t)|^2 + |c_b(t)|^2 = \cos^2\left(\frac{\omega}{2}t\right) + \sin^2\left(\frac{\omega}{2}t\right)\frac{\left[\omega_0^2 + \frac{4}{\hbar^2}|H'_{ba}|^2\right]}{\omega^2} = 1 \tag{16}$$

4. Griffith 9.9

   If we consider a ground state only, the density of states is given by the degeneracy. Thus the energy density will be $\rho_{g.s.}(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}$. Using the incoherent and unpolarized transition rate,

$$R_{a\to b} = \frac{\pi}{3\epsilon_0\hbar^2}|\wp|^2\rho(\omega_0) = \frac{|\wp|^2\omega_0^3}{3\epsilon_0\hbar\pi c^3} \equiv A \tag{17}$$

5. Griffith 9.17

   Let us consider an infinite potential well, with solutions $\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{i\frac{E_n}{\hbar}t}$, where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. Let us temporarily offset the Hamiltonian by a uniform potential $V_0(t)$ in the period $t : [0, T]$. We may solve the exact probability amplitudes from, $\dot{c}_m(t) = -\frac{i}{\hbar}\sum_n c_n H'_{mn}e^{i\omega_{mn}t}$, where $\omega_{mn} = \frac{E_m - E_n}{\hbar}$, $H'_{mn} =< \psi_m|H'|\psi_n >= V_0(t)\delta_{mn}$. Thus

$$\dot{c}_m(t) = -\frac{i}{\hbar}c_m V_0(t) \tag{18}$$

   which has the solution,

$$c_m(t) = c_m(0)e^{i\Phi} \tag{19}$$

   where the phase change is given by $\Phi = -\frac{1}{\hbar}\int_0^T dt'\, V_0(t')$. Using first-order perturbation theory, we find $c_m(t) = 0$, $m \neq N$, and $c_N(t) = 1 + i\Phi$. It is clear that this is the same answer as the exact result Taylor expanded to first-order with small phase.