PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #8

(1) Consider a ferromagnetic spin-S Ising model on a lattice of coordination number z. The Hamiltonian is

$$\hat{H} = -J \sum_{\langle ij \rangle} \sigma_i \, \sigma_j - \mu_0 H \sum_i \sigma_i \ ,$$

where $\sigma \in \{-S, -S+1, \ldots, +S\}$ with $2S \in \mathbb{Z}$.

(a) Find the mean field Hamiltonian $\hat{H}_{\text{MF}}$.

(b) Adimensionalize by setting $\theta \equiv k_{\text{B}} T / zJ$, $h \equiv \mu_0 H / zJ$, and $f \equiv F/NzJ$. Find the dimensionless free energy per site $f(m,h)$ for arbitrary S.

(c) Expand the free energy as

$$f(m,h) = f_0 + \tfrac{1}{2} a m^2 + \tfrac{1}{4} b m^4 - c h m + \mathcal{O}(h^2, hm^3, m^6)$$

and find the coefficients f_0, a, b, and c as functions of θ and S.

(d) Find the critical point $(\theta_{\text{c}}, h_{\text{c}})$.

(e) Find $m(\theta_{\text{c}}, h)$ to leading order in h.

(2) The Blume-Capel model is a $S = 1$ Ising model described by the Hamiltonian

$$\hat{H} = -\tfrac{1}{2} \sum_{i,j} J_{ij} \, S_i \, S_j + \Delta \sum_i S_i^2 \ ,$$

where $J_{ij} = J(\boldsymbol{R}_i - \boldsymbol{R}_j)$ and $S_i \in \{-1, 0, +1\}$. The mean field theory for this model is discussed in section 7.11 of the Lecture Notes, using the 'neglect of fluctuations' method. Consider instead a variational density matrix approach. Take $\varrho(S_1, \ldots, S_N) = \prod_i \tilde{\varrho}(S_i)$, where

$$\tilde{\varrho}(S) = \left(\frac{n+m}{2} \right) \delta_{S,+1} + (1-n) \, \delta_{S,0} + \left(\frac{n-m}{2} \right) \delta_{S,-1} \ .$$

(a) Find $\langle 1 \rangle$, $\langle S_i \rangle$, and $\langle S_i^2 \rangle$.

(b) Find $E = \text{Tr}\,(\varrho H)$.

(c) Find $S = -k_{\text{B}} \, \text{Tr}\,(\varrho \ln \varrho)$.

(d) Adimensionalizing by writing $\theta = k_{\text{B}} T / \hat{J}(0)$, $\delta = \Delta / \hat{J}(0)$, and $f = F / N \hat{J}(0)$, find the dimensionless free energy per site $f(m, n, \theta, \delta)$.

(e) Write down the mean field equations.

(f) Show that $m = 0$ always permits a solution to the mean field equations, and find $n(\theta, \delta)$ when $m = 0$.

(g) To find θ_c, set $m = 0$ but use both mean field equations. You should recover eqn. 7.322 of the Lecture Notes.

(h) Show that the equation for θ_c has two solutions for $\delta < \delta_*$ and no solutions for $\delta > \delta_*$, and find the value of δ_*.[1]

(i) Assume $m^2 \ll 1$ and solve for $n(m, \theta, \delta)$ using one of the mean field equations. Plug this into your result for part (d) and obtain an expansion of f in terms of powers of m^2 alone. Find the first order line. You may find it convenient to use `Mathematica` here.

[1]This problem has been corrected: (θ_*, δ_*) is not the tricritical point.