PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #9

(1) Consider a two-state Ising model, with an added quantum dash of flavor. You are invited to investigate the *transverse Ising model*, whose Hamiltonian is written

$$\hat{H} = -J \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - H \sum_i \sigma_i^z \ ,$$

where the σ_i^α are Pauli matrices:

$$\sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i \quad , \quad \sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i \ .$$

(a) Using the trial density matrix,

$$\varrho_i = \tfrac{1}{2} + \tfrac{1}{2}\, m_x\, \sigma_i^x + \tfrac{1}{2}\, m_z\, \sigma_i^z$$

on each site, compute the mean field free energy $F/N\hat{J}(0) \equiv f(\theta, h, m_x, m_z)$, where $\theta = k_B T/\hat{J}(0)$, and $h = H/\hat{J}(0)$. *Hint: Work in an eigenbasis when computing* $\mathsf{Tr}\,(\varrho \ln \varrho)$.

(b) Derive the mean field equations for m_x and m_z.

(c) Show that there is always a solution with $m_x = 0$, although it may not be the solution with the lowest free energy. What is $m_z(\theta, h)$ when $m_x = 0$?

(d) Show that $m_z = h$ for all solutions with $m_x \neq 0$.

(e) Show that for $\theta \leq 1$ there is a curve $h = h^*(\theta)$ below which $m_x \neq 0$, and along which m_x vanishes. That is, sketch the mean field phase diagram in the (θ, h) plane. Is the transition at $h = h^*(\theta)$ first order or second order?

(f) Sketch, on the same plot, the behavior of $m_x(\theta, h)$ and $m_z(\theta, h)$ as functions of the field h for fixed θ. Do this for $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$.