KIM GRIEST

# Solution to Final Problem 8

$\boxed{\text{FRW metric}}$

(a) $ds^2 = -dt^2 + R(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$

$d\chi = \frac{dr}{\sqrt{1-kr^2}} \Rightarrow \quad k=0 \quad d\chi = dr \Rightarrow \boxed{\chi = r} \Rightarrow \boxed{S(\chi) = r = \chi}$

$k=+1 \quad d\chi = \frac{dr}{\sqrt{1-r^2}} \Rightarrow \boxed{\chi = \arcsin r}$

$\Rightarrow r = \sin\chi$

$= S(\chi) = \sin\chi$

$k=-1 \quad d\chi = \frac{dr}{\sqrt{1+r^2}} \Rightarrow \chi = \text{arcsinh}\, r$

$\Rightarrow \boxed{r = S(\chi) = \sinh\chi}$

$$\boxed{ds^2 = -dt^2 + R(t)^2\left[d\chi^2 + S(\chi)^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]}$$

(b) $g_{tt} = -1, \quad g_{\chi\chi} = R(t)^2, \quad g_{\theta\theta} = R^2 S^2, \quad g_{\phi\phi} = R^2 S^2 \sin^2\theta$

(c) (d) $\boxed{\text{Find conserved quantities \& geodesic equations.}}$

Recall $\boxed{\frac{du_\alpha}{d\tau} = \frac{1}{2} g_{\mu\nu,\alpha} u^\mu u^\nu}$ where $\boxed{u^\mu = dx^\mu/d\tau}$

& $\boxed{u_\alpha = g_{\alpha\beta} u^\beta}$

$\boxed{\text{For } \phi \text{ component:}}$ $g_{\mu\nu,\phi} = 0 \Rightarrow \frac{du_\phi}{d\tau} = 0$

$\Rightarrow u_\phi = \text{constant}$ (conserved)

$u_\phi = g_{\phi\phi} u^\phi = R^2 S(\chi)^2 \sin^2\theta\, u^\phi$

Now at $\chi = 0$, $S(\chi) = 0$ so $u_\phi = 0$ at $\chi = 0$

Since $u_\phi$ is constant along the geodesic $u_\phi = 0$ everywhere

If $u_\phi = 0 \Rightarrow u^\phi = 0 \Rightarrow u^\phi = \frac{d\phi}{d\tau} = 0 \Rightarrow$ $\boxed{\phi = \text{constant along the geodesic}}$

$\boxed{\text{For } \theta \text{ component:}}$ only $g_{\phi\phi}$ depends on $\phi$

$\frac{du_\theta}{d\tau} = \frac{1}{2} g_{\phi\phi,\theta} u^\phi u^\phi = 2\sin\theta\cos\theta (u^\phi)^2 = 0$ since $u^\phi = 0$

$u_\theta$ = constant along geodesic

$u_\theta = g_{\theta\theta} u^\theta = R^2 S^2 u^\theta \Rightarrow u_\theta = 0$ at $\chi = 0 \Rightarrow u_\theta = 0$ along geodesic (conserved)

$u_\theta = 0 \Rightarrow u^\theta = 0 \Rightarrow \frac{d\theta}{d\tau} = 0 \Rightarrow$ $\boxed{\theta = \text{constant along geodesic}}$

PTO

$\boxed{\chi \text{ component}}$

$$\frac{du_\chi}{d\tau} = \frac{1}{2} g_{\mu\nu,\chi} u^\mu u^\nu$$

We know from before that $u^\theta = u^\phi = 0$

$$= \frac{1}{2} g_{00,\chi} u^0 u^0 + \frac{1}{2} g_{\chi\chi,\chi} u^\chi u^\chi + \frac{1}{2} g_{\theta\theta,\chi} u^\theta u^\theta + \frac{1}{2} g_{\phi\phi} u^\phi u^\phi$$

$\to 0$

$$= \frac{1}{2} R^2 \frac{dS^2}{d\chi} u^\theta u^\theta + \frac{1}{2} R^2 \sin^2\theta \frac{dS^2}{d\chi} u^\phi u^\phi$$

$= 0$ since $u^\theta = u^\phi = 0$

$\Rightarrow \frac{du_\chi}{d\tau} = 0 \Rightarrow u_\chi =$ constant along geodesic

$$u_\chi = g_{\chi\chi} u^\chi = \boxed{R^2 \frac{d\chi}{d\tau} = \text{constant}}$$

$$\Rightarrow \boxed{\frac{d\chi}{d\tau} = \frac{\text{constant}}{R^2}}$$

$\boxed{t \text{ component}}$ $u^\alpha u_\alpha = -1$ normalization

$$-u^t u_t + u^\chi u_\chi + u^\theta u_\theta + u^\phi u_\phi = -1$$

$$-\left(\frac{dt}{d\tau}\right)^2 + g^{\chi\chi} u_\chi u_\chi = -1$$

geodesic equation

$$-\left(\frac{dt}{d\tau}\right)^2 + \frac{\left(R^2 (d\chi/d\tau)\right)^2}{R^2} = -1 \Rightarrow \boxed{\left(\frac{dt}{d\tau}\right)^2 = 1 + R^2\left(\frac{d\chi}{d\tau}\right)^2}$$

(e) Interpret $\frac{d\chi}{d\tau} = \frac{\text{const}}{R^2}$ for $k=0$ case

For $k=0$, $\chi = r \Rightarrow \boxed{\frac{dr}{dt} = \frac{\text{const}}{R(t)^2}}$

If particle starts with peculiar velocity $v_p = \frac{dr}{dt}$

that velocity will reduce as $R(t)$ expands.