PHYS 2D
DISCUSSION SECTION

2012/5/23

Topics

- Uncertainty
- Tunneling

Wave Packet Review

- Particles are described by wave packets $\Psi(x,t)$
- The shape of the wave packet determines the probability distribution of the particle
- Particle can be found anywhere $|\Psi(x,t)|^2 \neq 0$
- A sharp spatial wave packet means it's very likely the particle will be found in a small region
- A sharp spatial wave packet will correspond to a wide wave packet in momentum space
- So the particle have a wide range of possible momenta

Uncertainty of Gaussian Wave Packet

- Consider a Gaussian wave packet in space
  - $\psi(x) = \left(\frac{a^2}{\pi}\right)^{\frac{1}{4}} e^{\frac{-(x-x_0)^2}{2a^2}}$
  - A bell-shaped wave packet centered at x_0
  - The width of the packet is proportional to a
- Expectation value of the particle's position

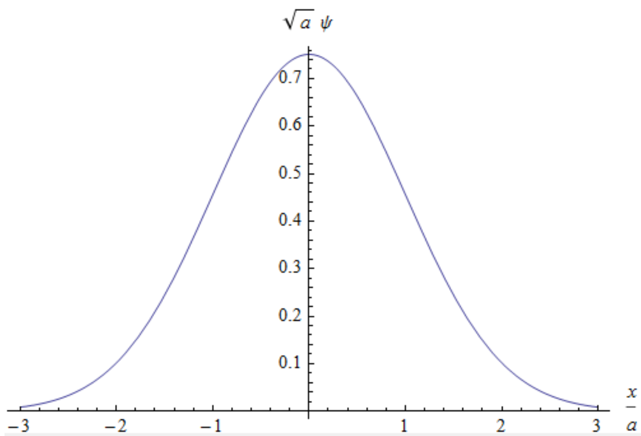
$$< x >= \int_{-\infty}^{\infty} x\, |\psi(x)|^2\, dx = x_0$$

- Average of displacement squared

$$< x^2 >= \int_{-\infty}^{\infty} x^2\, |\psi(x)|^2\, dx = \frac{a^2}{2} + x_0$$

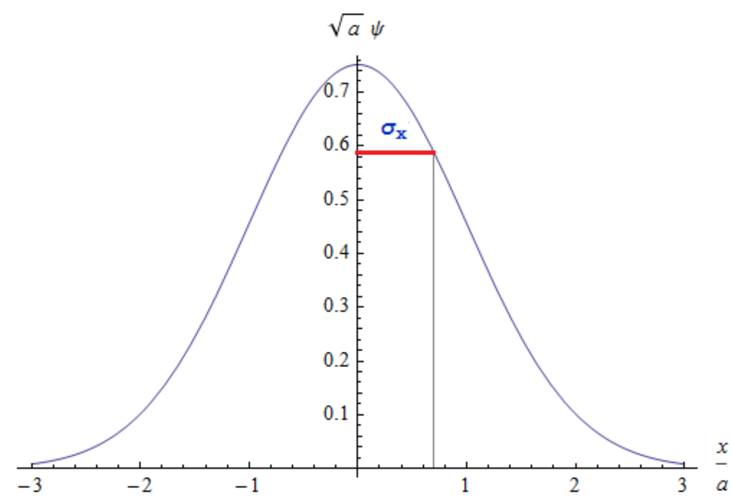
- RMS displacement from average

$$\sigma_x^2 \equiv < x^2 > - < x >^2$$



Meaning of Uncertainty Δx

- Set $x_0=0$ without loss of generality
- $<x>=0$, $<x^2>=a^2/2$
- $\sigma_x=a/2^{1/2}$
  - σ_x is the RMS displacement from average position
  - A measurement will likely give an average result of x_0, with a range on order $\pm\sigma_x$
  - Call σ_x the uncertainty Δx of the measurement of position x



Uncertainty in Momentum

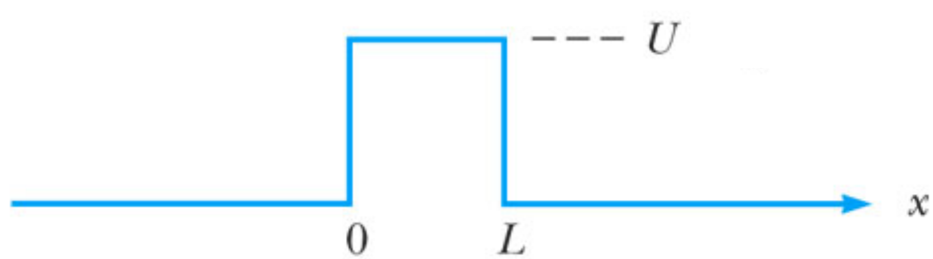
- What does this wave packet look like in k space?
  - $\psi(x) = \left(\frac{a^2}{\pi}\right)^{\frac{1}{4}} e^{\frac{-(x-x_0)^2}{2a^2}}$
  - By Fourier transform, $a(k)=C\exp(-a^2k^2/2)$
  - Also Gaussian, with $a \rightarrow 1/a$
  - By analogy with spatial wave, $\Delta k_x=\sigma_k=1/2^{1/2}a$
  - $p=\hbar k$, $\Delta p_x=\hbar/2^{1/2}a$
  - $\Delta p_x \Delta x=(\hbar/2^{1/2}a)(a/2^{1/2})=\hbar/2$
  - Recover uncertainty principle!

Uncertainty in General

- In general the wave packet is not Gaussian, but the idea for uncertainty is the same
- Uncertainty is a measure of the width of the wave packet
- It is also a measure of the size of the range of possible results from a measurement
- Which is to say it's the RMS displacement from the average position
- $(\Delta x)^2 = \sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$

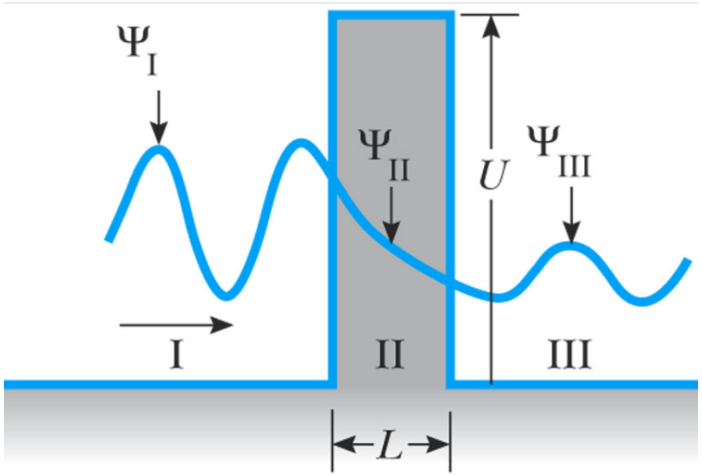
The Tunneling Effect

- What is tunneling?
- A particle with energy E hits a potential wall U
- Classically if E<U the particle bounces back
- In QM a portion of the wave packet passes through



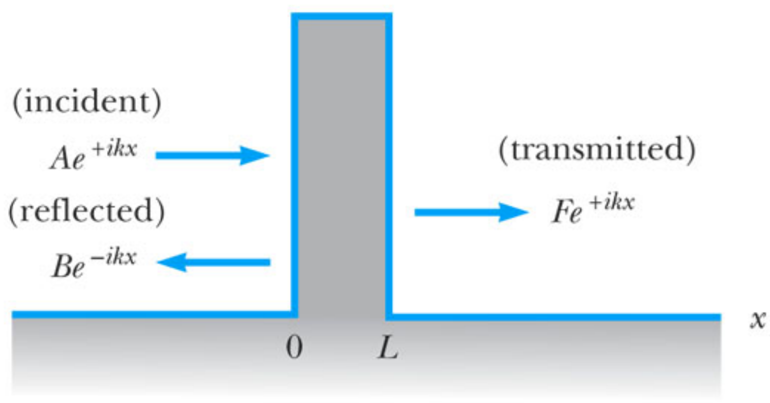
Scattering Picture

- To find out what's going on, find the wave function for the particle
- Solve Schrodinger's equation in 3 regions
- Think of the particle as coming from the left
- When it hits the barrier, part of it will reflect & part of it will go through
- We want to know how much goes through and how much is reflected



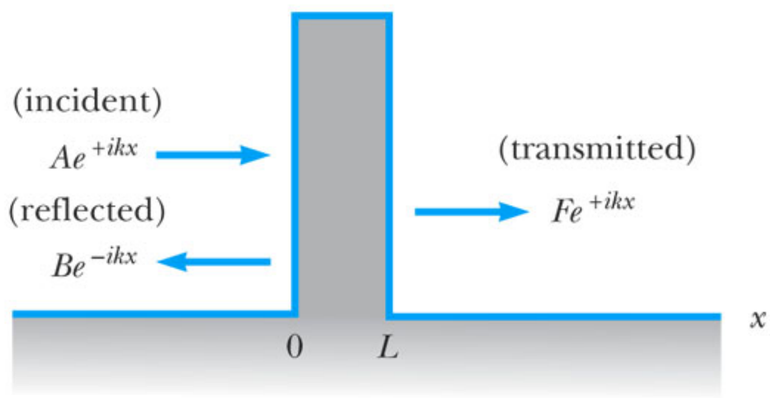
Form of the Wave Function

- When E>U, we know the solution for ψ(x) is sin(kx) & cos(kx), or exp(ikx)
- A plane wave going to the right is exp[i(kx-ωt)], while one going to the left is exp[i(-kx-ωt)]
- In barrier region E<U, the solution is exp(±αx-iωt)
- Drop exp(-iωt)
- For x>L we assume no incoming wave from the right



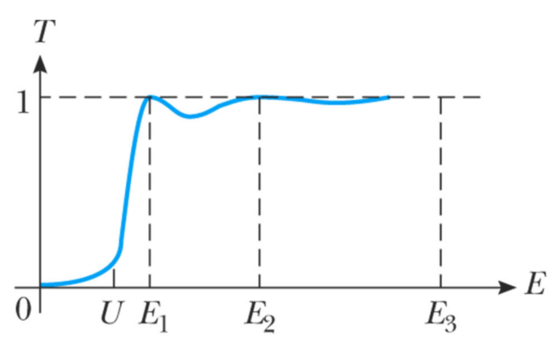
Solving for ψ(x)

- Spatial wave functions in regions I, II, III
- In region II, $\psi(x)=C\exp(-\alpha x)+D\exp(\alpha x)$
- Require ψ & $d\psi/dx$ be continuous at boundaries $x=0$ & $x=L$
- Solve for the coefficients, 5 unknowns with 4 equations, 1 unknown represents overall factor



Tunneling Coefficients

- Define T=|F/A|², R=|B/A|²
- $$T(E) = \left[1 + \frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L)\right]^{-1}$$ $$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$$
- When E<U, most of the wave is reflected
- When E>U, most of the wave goes through, T oscillates below 1
- T=1 happens when reflected wave from x=0 & x=L cancel



- Questions?