6-44. (a) For $x > 0$, $\hbar^2 k_2^2 / 2m + V_0 = E = \hbar^2 k_1^2 / 2m = 2V_0$

So, $k_2 = \ 2mV_0\ ^{1/2}/\hbar$. Because $k_1 = \ 4mV_0\ ^{1/2}/\hbar$, then $k_2 = k_1/\sqrt{2}$

(b) $R = \ k_1 - k_2\ ^2 / \ k_1 + k_2\ ^2$ (Equation 6-68)

$= \ 1 - 1/\sqrt{2}\ ^2 / \ 1 + 1/\sqrt{2}\ ^2 = 0.0294$, or 2.94% of the incident particles are reflected.

(c) $T = 1 - R = 1 - 0.0294 = 0.971$

(d) 97.1% of the particles, or $0.971 \times 10^6 = 9.71 \times 10^5$, continue past the step in the $+x$ direction. Classically, 100% would continue on.

6-45 (a) Equation 6-76: $T \approx 16 \frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2\alpha a}$ where $\alpha = 2\sqrt{2m_p(V_0 - E)}/\hbar$

and $a =$ barrier width.

$$-2\alpha a = -2\left[\sqrt{2(938\,\mathrm{MeV/c^2})(50-44)\mathrm{MeV}} / 6.58\times10^{-22}\,\mathrm{MeV\cdot s}\right]\times10^{-15} = -1.075$$

$$T \approx 16\frac{44\,\mathrm{MeV}}{50\,\mathrm{MeV}}\left(1 - \frac{44\,\mathrm{MeV}}{50\,\mathrm{MeV}}\right)e^{-1.075}$$

$$T \approx 0.577$$

(b) decay rate $\approx N \times T$ where

$$N = \frac{v_{proton}}{2R} = \left[\frac{2\times 44\,\mathrm{MeV}\times1.60\times10^{-13}\,\mathrm{J/MeV}}{1.67\times10^{-27}\,\mathrm{kg}}\right]^{1/2}\times\frac{1}{2\times10^{-15}\,\mathrm{m}} = 4.59\times10^{22}\,\mathrm{s}^{-1}$$

$$\text{decay rate} \approx 0.577\times4.59\times10^{22}\,\mathrm{s}^{-1} = 2.65\times10^{22}\,\mathrm{s}^{-1}$$

(c) In the expression for T, $e^{-1.075} \Rightarrow e^{-2.150}$, and so $T \approx 0.577 \Rightarrow T \approx 0.197$. The decay rate then becomes $9.05\times10^{21}\,\mathrm{s}^{-1}$, a factor of $0.34\times$ the original value.

6-46. (a) For $x > 0$, $\hbar^2 k_2^2 / 2m - V_0 = E = \hbar^2 k_1^2 / 2m = 2V_0$

So, $k_2 = \ 6mV_0\ ^{1/2}/\hbar$. Because $k_1 = \ 4mV_0\ ^{1/2}/\hbar$, then $k_2 = \sqrt{3/2}\,k_1$

(b) $R = (k_1 - k_2)^2 / (k_1 + k_2)^2$

$$R = (k_1 - k_2)^2 / (k_1 + k_2)^2 = \left(1 - \sqrt{3/2}\right)^2 / \left(1 + \sqrt{3/2}\right)^2 = 0.0102$$

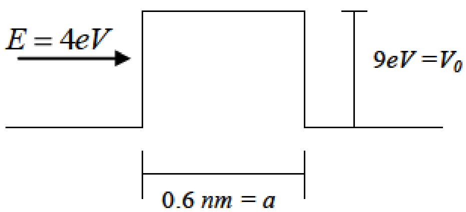
Or 1.02% are reflected at $x = 0$.

(c) $T = 1 - R = 1 - 0.0102 = 0.99$

(d) 99% of the particles, or $0.99 \times 10^6 = 9.9 \times 10^5$, continue in the $+x$ direction.

Classically, 100% would continue on.

6-47. (a)



$$\alpha = \sqrt{2m(V_0 - E)}/\hbar$$

$$= \sqrt{2(0.511 \times 10^6 eV/c^2)(5eV)}/\hbar$$

$$= \sqrt{5.11 \times 10^6 eV}\frac{eV}{c}/\hbar$$

$$= \frac{2260eV}{197.3eV \bullet nm} = 11.46nm^{-1}$$

and $\alpha a = 0.6nm \times 11.46nm^{-1} = 6.87$

Since αa is not $\ll 1$, use Equation 6-75:

The transmitted fraction

$$T = \left[1 + \frac{\sinh^2 \alpha a}{4(E/V_0)(1 - E/V_0)}\right]^{-1} = \left[1 + \left(\frac{81}{80}\right)\sinh^2(6.87)\right]^{-1}$$

Recall that $\sinh x = (e^x - e^{-x})/2$,

$$T = \left[1 + \frac{81}{80}\left(\frac{e^{6.87} - e^{-6.87}}{2}\right)^2\right]^{-1} = 4.3 \times 10^{-6}$$ is the transmitted fraction.

(b) Noting that the size of T is controlled by αa through the $\sinh^2 \alpha a$ and increasing T implies increasing E. Trying a few values, selecting $E = 4.5eV$ yields $T = 8.7 \times 10^{-6}$ or approximately twice the value in part (a).

6-50. Using Equation 6-76,

$$T \approx 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\alpha a} \text{ where } E = 2.0eV,\ V_0 = 6.5eV, \text{ and } a = 0.5nm.$$

$$T \approx 16\left(\frac{2.0}{6.5}\right)\left(1-\frac{2.0}{6.5}\right)e^{-2\ 10.87\ \ 0.5} \approx 6.5\times10^{-5}$$ (Equation 6-75 yields $T = 6.6\times10^{-5}$.)

6-51. $R = \dfrac{k_1-k_2\ ^2}{k_1+k_2\ ^2}$ and $T = 1-R$ (Equations 6-68 and 6-70)

(a) For protons:

$$k_1 = \sqrt{2mc^2E}\,/\,\hbar c = \sqrt{2\ \ 938MeV\ \ \ 40MeV}\,/197.3MeV\bullet fm = 1.388$$

$$k_2 = \sqrt{2mc^2\ \ E-V_0}\,/\,\hbar c = \sqrt{2\ \ 938MeV\ \ \ 10MeV}\,/197.3MeV\bullet fm = 0.694$$

$$R = \left(\frac{1.388-0.694}{1.388+0.694}\right)^2 = \left(\frac{0.694}{2.082}\right)^2 = 0.111 \quad \text{And } T = 1-R = 0.889$$

(b) For electrons:

$$k_1 = 1.388\left(\frac{0.511}{938}\right)^{1/2} = 0.0324 \qquad k_2 = 0.694\left(\frac{0.511}{938}\right)^{1/2} = 0.0162$$

$$R = \left(\frac{0.0324-0.0162}{0.0324+0.0162}\right)^2 = 0.111 \quad \text{And } T = 1-R = 0.889$$

No, the mass of the particle is not a factor. (We might have noticed that $\sqrt{m}$ could be canceled from each term.

6-56. (a) The requirement is that $\psi^2 \; x \; = \psi^2 \; -x \; = \psi \; -x \; \psi \; -x$. This can only be true if:

$\psi \; -x \; = \psi \; x$ or $\psi \; -x \; = -\psi \; x$.

(b) Writing the Schrödinger equation in the form $\dfrac{d^2\psi}{dx^2} = -\dfrac{2mE}{\hbar^2}\psi$, the general solutions of this 2nd order differential equation are: $\psi \; x \; = A\sin kx$ and $\psi \; \mathrm{x} \; = A\cos kx$ where $k = \sqrt{2mE}/\hbar$. Because the boundaries of the box are at $x = \pm L/2$, both solutions are allowed (unlike the treatment in the text where one boundary was at $x = 0$). Still, the solutions are all zero at $x = \pm L/2$ provided that an integral number of half wavelengths fit between $x = -L/2$ and $x = +L/2$. This will occur for:

$\psi_n \; x \; = \; 2/L \; ^{1/2} \cos n\pi x/L$ when $n = 1, 3, 5, \cdots$. And for

$\psi_n \; x \; = \; 2/L \; ^{1/2} \sin n\pi x/L$ when $n = 2, 4, 6, \cdots$.

The solutions are alternately even and odd.

(c) The allowed energies are: $E = \hbar^2 k^2 / 2m = \hbar^2 \; n\pi / L \; ^2 / 2m = n^2 h^2 / 8mL^2$.

6-57. $\psi_0 = Ae^{-x^2/2L^2}$

(a) $\dfrac{d\psi_0}{dx} = \; -x/L^2 \; Ae^{-x^2/2L^2}$ and $\psi_1 = L\dfrac{d\psi_0}{dx} = L \; -x/L^2 \; Ae^{-x^2/2L^2} \; = \; -x/L \; \psi_0$

So, $\dfrac{d\psi_1}{dx} = - \; 1/L \; \psi_0 - \; x/L \; d\psi_0/dx$

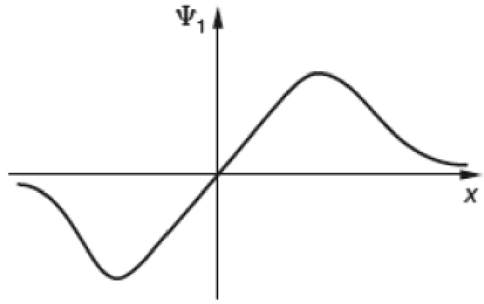
And $\dfrac{d^2\psi_1}{dx^2} = - \; 1/L \; d\psi_0/dx - \; 1/L \; d\psi_0/dx - \; x/L \; d^2\psi_0/dx^2$

$$= \; 2x/L^3 \; \psi_0 + \; x/L^3 \; \psi_0 + \; x^3/L^5 \; \psi_0$$

Recalling from Problem 6-3 that $V \; x \; = \hbar^2 x^2 / 2mL^4$, the Schrödinger equation becomes $-\hbar^2/2m \;\; 3m/L^3 + x^3/L^5 \; \psi_0 + \; \hbar^2 x^3 / 2mL^5 \; \psi_0 = E \; -x/L \; \psi_0$ or, simplifying: $-3\hbar^2 x / 2mL^3 \; \psi_0 = E \; -x/L \; \psi_0$. Thus, choosing E appropriately will make ψ_1 a solution.

(b) We see from (a) that $E = 3\hbar^2 / 2mL^2$, or three times the ground state energy.

(c) ψ_1 plotted looks as below. The single node indicates that ψ_1 is the first excited state. (The energy value in [b] would also tell us that.)



6-58. $\langle x^2 \rangle = \int_0^L \frac{2}{L} x^2 \sin^2 \frac{n\pi x}{L} dx$ Letting $u = n\pi x / L$, $du = (n\pi / L)\, dx$

$$\langle x^2 \rangle = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left(\frac{L}{n\pi}\right) \int_0^{n\pi} u^2 \sin^2 u\, du$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}\right]_0^{n\pi}$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \left[\frac{(n\pi)^3}{6} - 0 - \frac{n\pi}{4} - 0\right] = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$$

6-60. (a) For $\Psi(x,t) = A\sin(kx - \omega t)$

$\frac{d^2\Psi}{dx^2} = -k^2\Psi$ and $\frac{\partial \Psi}{\partial t} = -\omega A \cos(kx - \omega t)$ so the Schrödinger equation becomes:

$$-\frac{\hbar^2 k^2}{2m} A\sin(kx - \omega t) + V(x) A\sin(kx - \omega t) = -i\hbar\omega\cos(kx - \omega t)$$

Because the *sin* and *cos* are not proportional, this Ψ cannot be a solution. Similarly, for $\Psi(x,t) = A\cos(kx - \omega t)$, there are no solutions.

(b) For $\Psi(x,t) = A[\cos(kx - \omega t) + i\sin(kx - \omega t)] = Ae^{i(kx-\omega t)}$, we have that

$\frac{d^2\Psi}{dx^2} = -k^2\Psi$ and $\frac{\partial \Psi}{\partial t} = -i\omega\Psi$. And the Schrödinger equation becomes:

$$-\frac{\hbar^2 k^2}{2m}\Psi + V(x)\Psi = -\hbar\omega\Psi \text{ for } \hbar\omega = \hbar^2 k^2 / 2m + V.$$