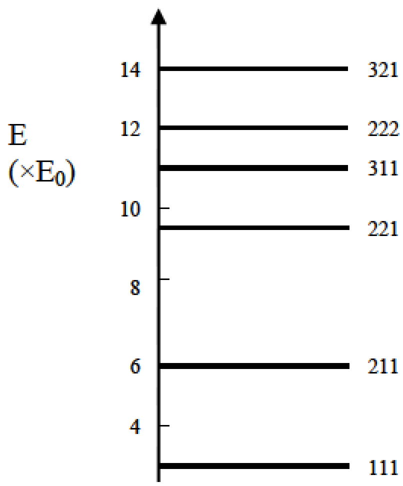
7-1. $E_{n_1 n_2 n_3} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right)$ (Equation 7-4)

$$E_{311} = \frac{\hbar^2\pi^2}{2mL^2}\left(3^2 + 1^2 + 1^2\right) = 11E_0 \quad \text{where } E_0 = \frac{\hbar^2\pi^2}{2mL^2}$$

$$E_{222} = E_0\left(2^2 + 2^2 + 2^2\right) = 12E_0 \quad \text{and} \quad E_{321} = E_0\left(3^2 + 2^2 + 1^2\right) = 14E_0$$

The 1st, 2nd, 3rd, and 5th excited states are degenerate.



7-2. $E_{n_1 n_2 n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right) = \frac{\hbar^2\pi^2}{2mL_1^2}\left(n_1^2 + \frac{n_2^2}{4} + \frac{n_3^2}{9}\right)$ (Equation 7-5)

$n_1 = n_2 = n_3 = 1$ is the lowest energy level.

$$E_{111} = E_0\left(1 + 1/4 + 1/9\right) = 1.361E_0 \quad \text{where } E_0 = \frac{\hbar^2\pi^2}{2mL_1^2}$$

The next nine levels are, increasing order,

n_1	n_2	n_3	$E\left(\times E_0\right)$
1	1	2	1.694
1	2	1	2.111
1	1	3	2.250
1	2	2	2.444
1	2	3	3.000
1	1	4	3.028
1	3	1	3.360
1	3	2	3.472
1	2	4	3.778

7-3. (a) $\psi_{n_1 n_2 n_3}(x,y,z) = A\cos\frac{n_1\pi x}{L}\sin\frac{n_2\pi y}{L}\sin\frac{n_3\pi z}{L}$

(b) They are identical. The location of the coordinate origin does not affect the energy level structure.

7-4. $\psi_{111}(x,y,z) = A\sin\frac{\pi x}{L_1}\sin\frac{\pi y}{2L_1}\sin\frac{\pi z}{3L_1}$ $\psi_{112}(x,y,z) = A\sin\frac{\pi x}{L_1}\sin\frac{\pi y}{2L_1}\sin\frac{2\pi z}{3L_1}$

$\psi_{121}(x,y,z) = A\sin\frac{\pi x}{L_1}\sin\frac{\pi y}{L_1}\sin\frac{\pi z}{3L_1}$ $\psi_{122}(x,y,z) = A\sin\frac{\pi x}{L_1}\sin\frac{\pi y}{L_1}\sin\frac{2\pi z}{3L_1}$

$\psi_{113}(x,y,z) = A\sin\frac{\pi x}{L_1}\sin\frac{\pi y}{2L_1}\sin\frac{\pi z}{L_1}$

7-7. $$E_0 = \frac{\hbar^2\pi^2}{2mL^2} = \frac{\left(1.055\times10^{-34}J\bullet s\right)^2\pi^2}{2\left(9.11\times10^{-31}kg\right)\left(0.10\times10^{-9}m\right)^2\left(1.609\times10^{-19}J/eV\right)} = 37.68eV$$

$$E_{311} - E_{111} = \Delta E = 11E_0 - 3E_0 = 8E_0 = 301eV$$

$$E_{222} - E_{111} = \Delta E = 12E_0 - 3E_0 = 9E_0 = 339eV$$

$$E_{321} - E_{111} = \Delta E = 14E_0 - 3E_0 = 11E_0 = 415eV$$

7-8. (a) Adapting Equation 7-3 to two dimensions (i.e., setting $k_3 = 0$), we have

$$\psi_{n_1 n_2} = A\sin\frac{n_1\pi x}{L}\sin\frac{n_2\pi y}{L}$$

(b) From Equation 7-5, $E_{n_1 n_2} = \dfrac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2\right)$

(c) The lowest energy degenerate states have quantum numbers $n_1 = 1$, $n_2 = 2$, and $n_1 = 2$, $n_2 = 1$.

7-9. (a) For $n = 3$, $\ell = 0, 1, 2$

(b) For $\ell = 0, m = 0$. For $\ell = 1, m = -1, 0, +1$. For $\ell = 2, m = -2, -1, 0, +1, +2$.

(c) There are nine different m-states, each with two spin states, for a total of 18 states for $n = 3$.

7-10. (a) For $\ell = 4$

$$L = \sqrt{\ell(\ell+1)}\,\hbar = \sqrt{4(5)}\,\hbar = \sqrt{20}\,\hbar$$

$$m_\ell = 4\hbar$$

$$\theta_{\min} = \cos^{-1}\frac{4}{\sqrt{20}} \quad \rightarrow \quad \theta_{\min} = 26.6°$$

(b) For $\ell = 2$

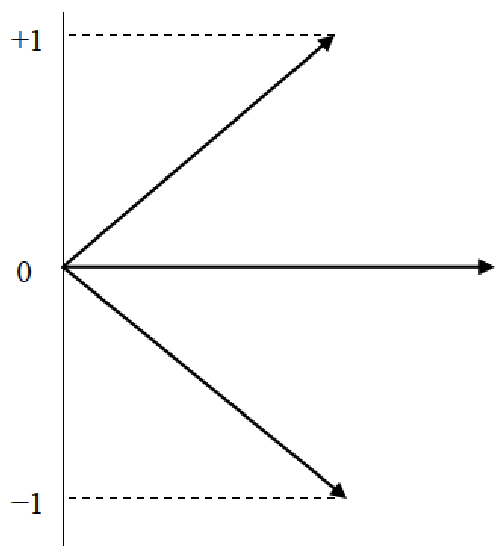
$$L = \sqrt{6}\,\hbar \qquad m_\ell = 2\hbar$$

$$\theta_{\min} = \cos^{-1}\frac{2}{\sqrt{6}} \quad \rightarrow \quad \theta_{\min} = 35.3°$$

7-12. (a)

$\ell = 1$

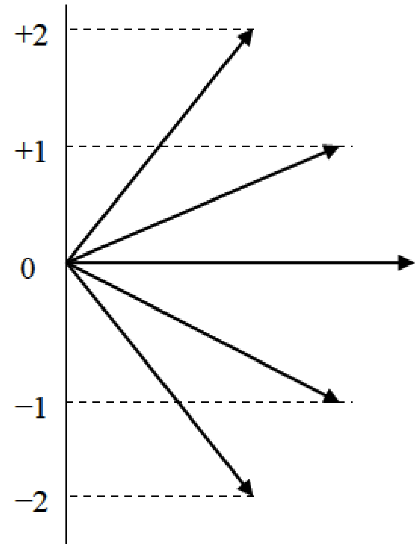
$|\mathbf{L}| = \sqrt{2}\hbar$



(b)

$\ell = 2$

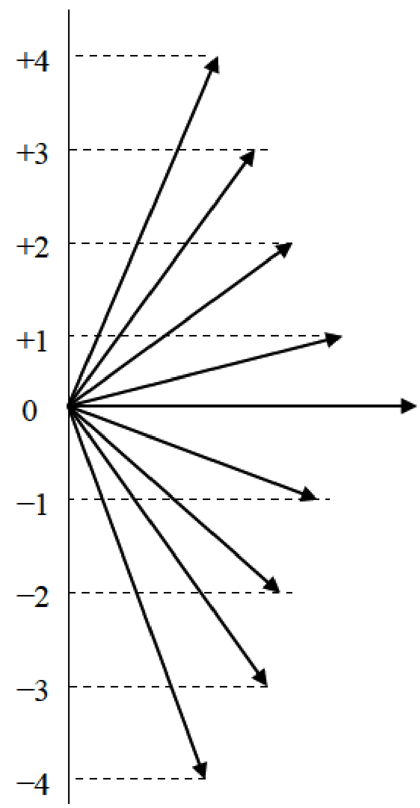
$|\mathbf{L}| = \sqrt{6}\hbar$



(c)

$\ell = 4$

$|\mathbf{L}| = \sqrt{20}\hbar$



(d) $|\boldsymbol{L}| = \sqrt{\ell(\ell+1)}\,\hbar$ (See diagrams above.)

7-13. $L^2 = L_x^2 + L_y^2 + L_z^2 \rightarrow L_x^2 + L_y^2 = L^2 - L_z^2 = \ell(\ell+1)\hbar^2 - (m\hbar)^2 = (6 - m^2)\hbar^2$

(a) $\left(L_x^2+L_y^2\right)_{\text{min}}=\left(6-2^2\right)\hbar^2=2\hbar^2$

(b) $\left(L_x^2+L_y^2\right)_{\text{max}}=\left(6-0^2\right)\hbar^2=6\hbar^2$

(c) $L_x^2+L_y^2=(6-1)\hbar^2=5\hbar^2$ $\quad L_x$ and L_y cannot be determined separately.

(d) $n=3$

7-15. $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$ $\qquad \dfrac{d\boldsymbol{L}}{dt}=\dfrac{d\boldsymbol{r}}{dt}\times\boldsymbol{p}+\boldsymbol{r}\times\dfrac{d\boldsymbol{p}}{dt}$

$\dfrac{d\boldsymbol{r}}{dt}\times\boldsymbol{p}=\boldsymbol{v}\times m\boldsymbol{v}=m\boldsymbol{v}\times\boldsymbol{v}=0$ and $\boldsymbol{r}\times\dfrac{d\boldsymbol{p}}{dt}=\boldsymbol{r}\times\boldsymbol{F}$. Since for $V=V(r)$, i.e., central forces,

$\boldsymbol{F}$ is parallel to $\boldsymbol{r}$, then $\boldsymbol{r}\times\boldsymbol{F}=0$ and $\dfrac{d\boldsymbol{L}}{dt}=0$

7-16. (a) For $\ell=3$, $n=4, 5, 6, \ldots$ and $m=-3, -2, -1, 0, 1, 2, 3$

(b) For $\ell=4$, $n=5, 6, 7, \ldots$ and $m=-4, -3, -2, -1, 0, 1, 2, ,3 ,4$

(c) For $\ell=0$, $n=1$ and $m=0$

(d) The energy depends only on n. The minimum in each case is:

$E_4=-13.6eV/n^2=-13.6eV/4^2=-0.85eV$

$E_5=-13.6eV/5^2=-0.54eV$

$E_1=-13.6eV$

7-17. (a) $6f$ state: $n=6, \ell=3$

(b) $E_6=-13.6eV/n^2=-13.6eV/6^2=-0.38eV$

(c) $L=\sqrt{\ell(\ell+1)}\,\hbar=\sqrt{3(3+1)}\,\hbar=\sqrt{12}\,\hbar=3.65\times10^{-34}J\bullet s$

(d) $L_z=m\hbar$ $\qquad L_z=-3\hbar, -2\hbar, -1\hbar, 0, 1\hbar, 2\hbar, 3\hbar$

7-21. (a) For the ground state, $P(r)\Delta r = \psi^2\left(4\pi r^2\right)\Delta r = \dfrac{4r^2}{a_0^3}e^{-2r/a_0}\Delta r$

For $\Delta r = 0.03a_0$, at $r = a_0$ we have $P(r)\Delta r = \dfrac{4a_0^2}{a_0^3}e^{-2}\left(0.03a_0\right) = 0.0162$

(b) For $\Delta r = 0.03a_0$, at $r = 2a_0$ we have $P(r)\Delta r = \dfrac{4\left(2a_0\right)^2}{a_0^3}e^{-4}\left(0.03a_0\right) = 0.0088$

7-22. $P(r) = Cr^2e^{-2Zr/a_0}$ For $P(r)$ to be a maximum,

$$\frac{dP}{dt} = C\left[r^2\left(-\frac{2Z}{a_0}\right)e^{-2Zr/a_0} + 2re^{-2Zr/a_0}\right] = 0 \rightarrow C\times\frac{2Zr}{a_0}\left(\frac{a_0}{Z} - r\right)e^{-2Zr/a_0} = 0$$

This condition is satisfied with $r = 0$ or $r = a_0/Z$. For $r = 0$, $P(r) = 0$ so the maximum $P(r)$ occurs for $r = a_0/Z$.

7-27. For the most likely value of r, $P(r)$ is a maximum, which requires that (see Problem 7-25)

$$\frac{dP}{dr} = A\cos^2\theta\left[r^4\left(-\frac{Z}{a_0}\right)e^{-Zr/a_0} + 4r^3e^{-Zr/a_0}\right] = 0$$

For hydrogen $Z = 1$ and $A\cos^2\theta\left(r^3/a_0\right)\left(4a_0 - r\right)e^{-r/a_0} = 0$. This is satisfied for $r = 0$ and $r = 4a_0$. For $r = 0$, $P(r) = 0$ so the maximum $P(r)$ occurs for $r = 4a_0$.

7-65. $\psi_{100} = \dfrac{1}{\sqrt{\pi}}\left(\dfrac{Z}{a_0}\right)^{3/2}e^{-Zr/a_0}$ (Equations 7-30 and 7-31)

$P(r) = 4\pi r^2\psi_{100}^*\psi_{100}$ (Equation 7-32)

$$= 4\pi r^2\frac{Z^3}{\pi a_0^3}e^{-Zr/a_0} = \frac{4Z^3}{a_0^3}r^2e^{-2Zr/a_0}$$

$$\langle r\rangle = \int_0^\infty rP(r)\,dr = \int_0^\infty \frac{4Z^3}{a_0^3}r^3e^{-2Zr/a_0}\,dr$$

$$= \frac{a_0}{4Z}\int_0^\infty\left(\frac{2Zr}{a_0}\right)^3 e^{-2Zr/a_0}\,d\left(2Zr/a_0\right) = \frac{a_0}{4Z}\times 3! = \frac{3a_0}{2Z}$$

7-68. $\theta_{\min} = \cos^{-1}\left[m_\ell \hbar / \sqrt{\ell(\ell+1)}\hbar\right]$ with $m_\ell = \ell$.

$\cos\theta_{\min} = \ell\sqrt{\ell(\ell+1)}$. Thus, $\cos^2\theta_{\min} = \ell^2/\left[\ell(\ell+1)\right] = 1 - \sin^2\theta_{\min}$

or, $\sin^2\theta_{\min} = 1 - \dfrac{\ell^2}{\ell(\ell+1)} = \dfrac{\ell(\ell+1)-\ell^2}{\ell(\ell+1)} = \dfrac{\ell^2+\ell-\ell^2}{\ell(\ell+1)}$

And, $\sin\theta_{\min} = \left(\dfrac{1}{\ell+1}\right)^{1/2}$ For large ℓ, $\theta_{\min}$ is small.

Then $\sin\theta_{\min} \approx \theta_{\min} = \left(\dfrac{1}{\ell+1}\right)^{1/2} \approx \dfrac{1}{(\ell)^{1/2}}$

7-70. $P(r) = \dfrac{4Z^3}{a_0^3} r^2 e^{-2Zr/a_0}$ (See Problem 7-65)

For hydrogen, $Z = 1$ and at the edge of the proton $r = R_0 = 10^{-15}m$. At that point, the exponential factor in P(r) has decreased to:

$$e^{-2R_0/a_0} = e^{-2(10^{-15})/(0.529\times10^{-10}m)} = e^{-(3.78\times10^{-5})} \approx 1 - 3.78\times10^{-5} \approx 1$$

Thus, the probability of the electron in the hydrogen ground state being inside the nucleus, to better than four figures, is:

$$P(r) = \frac{4r^2}{a_0^3} \qquad P = \int_0^{r_0} P(r)dr = \int_0^{R_0} \frac{4r^2}{a_0^3} = \frac{4}{a_0^3}\int_0^{R_0} r^2 dr = \frac{4}{a_0^3}\frac{r^3}{3}\bigg|_0^{R_0}$$

$$= \frac{4}{a_0^3}\left(\frac{R_0^3}{3}\right) = \frac{4\left(10^{-15}m\right)^3}{3\left(0.529\times10^{-10}m\right)^3} = 9.0\times10^{-15}$$

7-72. (a) Substituting $\psi(r,\theta)$ into Equation 7-9 and carrying out the indicated operations yields (eventually)

$$-\frac{\hbar^2}{2\mu}\psi(r,\theta)\left[2/r^2 - 1/4a_0^2\right] - \frac{\hbar^2}{2\mu}\psi(r,\theta)\left(-2/r^2\right) + V\psi(r,\theta) = E\psi(r,\theta)$$

Canceling $\psi(r,\theta)$ and recalling that $r^2 = 4a_0^2$ (because ψ given is for $n = 2$) we have $-\frac{\hbar^2}{2\mu}\left(-1/4a_0^2\right) + v = E$

The circumference of the $n = 2$ orbit is: $C = 2\pi\left(4a_0\right) = 2\lambda \rightarrow a_0 = \lambda/4\pi = 1/2k$.

Thus, $-\frac{\hbar^2}{2\mu}\left(-\frac{1}{4/4k^2}\right) + V = E \;\rightarrow\; \frac{\hbar^2 k^2}{2\mu} + V = E$

(b) or $\frac{p^2}{2m} + v = E$ and Equation 7-9 is satisfied.

$$\int_0^\infty \psi^2 dx = \int A^2 \left(\frac{r}{a_0}\right)^2 e^{-r/a_0} \cos^2\theta r^2 \sin\theta dr d\theta d\phi = 1$$

$$A^2 \int_0^\infty \left(\frac{r}{a_0}\right)^2 e^{-r/a_0} r^2 dr \int_0^\pi \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi = 1$$

Integrating (see Problem 7-23),

$$A^2\left(6a_0^3\right)(2/3)(2\pi) = 1$$

$$A^2 = 1/8a_0^3\pi \rightarrow A = \sqrt{1/8a_0^3\pi}$$