Physics 2D — Final Solutions — Winter 2012

Problem 1

Classically, $\boxed{M = m_1 + m_2}$. Momentum conservation gives:

$$m_1 u - m_2 u = M u' \Rightarrow \boxed{u' = \frac{m_1 - m_2}{m_1 + m_2} u} \text{ (a)}$$

(b) Relativistically:

Momentum conservation: $\gamma m_1 u - \gamma m_2 u = \gamma' M u'$

Energy conservation: $\gamma m_1 c^2 + \gamma m_2 c^2 = \gamma' M c^2 \Rightarrow$

$$\Rightarrow \gamma' M = \gamma (m_1 + m_2) \Rightarrow$$

Substituting in momentum conservation eq.

$$\gamma (m_1 - m_2) u = \gamma' M u' = \gamma (m_1 + m_2) u'$$

$$\Rightarrow \boxed{u' = \frac{m_1 - m_2}{m_1 + m_2} u} \text{ (b)}$$

it is <u>same</u> as value found classically.

(c) From energy equation,

$$\boxed{M = \frac{\gamma}{\gamma'} (m_1 + m_2)}$$

$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$, $\gamma' = \frac{1}{\sqrt{1 - u'^2/c^2}}$; since $u' < u \Rightarrow \gamma' < \gamma \Rightarrow$

$$\Rightarrow \frac{\gamma}{\gamma'} > 1 \Rightarrow \boxed{M > m_1 + m_2}$$

because in an inelastic collision, kinetic energy gets transformed into mass.

Problem 2

(a) $\lambda_m T = \dfrac{hc}{4.96 k_B} \Rightarrow \lambda_0 = \dfrac{12{,}400 \times 11{,}600\,\mathring{A}}{4.96 \times 5000} = 5800\,\mathring{A}$

$$\boxed{\lambda_0 = 5800\,\mathring{A}}$$

(b) if $T = 5000°K$, $T' = 10{,}000°K = 2T$, we have:

$$\frac{hc}{\lambda_0 k_B T} = 4.96, \quad \frac{hc}{\lambda_0 k_B T'} = \frac{4.96}{2} = 2.48$$

Power is proportional to $\dfrac{1}{e^{hc/\lambda k_B T} - 1}$; so

$$\frac{\text{power}(T')}{\text{power}(T)} = \frac{e^{4.96} - 1}{e^{2.48} - 1} = \frac{141.59}{10.94} = 12.94$$

$$\boxed{\text{So power \underline{increases} by factor } 12.94}$$

(c) Total power is proportional to T^4 (Stefan law)

T increases by factor of 2 $\Rightarrow$ $\boxed{\text{total power increases by factor } 16}$

(d) At very large λ

$$\frac{1}{e^{hc/\lambda kT} - 1} \approx \frac{1}{1 + \frac{hc}{\lambda kT} - 1} = \frac{\lambda k T}{hc} \qquad (\text{using } e^x \approx 1 + x)$$

so power is proportional to T $\Rightarrow$

$$\boxed{\text{power increases by factor of 2 at wavelengths} \gg \lambda_0}$$

Problem 3

The ions are in their ground state, so $n=1$ for the electron.

$E_1 = -E_0 Z^2$ with $E_0 = 13.6\,eV$. To ionize, photon has to have

energy $hf = \frac{hc}{\lambda} = E_0 Z^2 \Rightarrow \lambda = \frac{hc}{E_0 Z^2} = \frac{12{,}400\,eV\text{Å}}{13.6\,eV \cdot Z^2}$

$\Rightarrow \lambda = \frac{911.76\,\text{Å}}{Z^2}$. So to ionize atoms with $Z = 1, 2, 3, 4, \ldots$

requires $\lambda = 911\,\text{Å}, 228\,\text{Å}, 101\,\text{Å}, 57\,\text{Å}, \ldots$

Here, radiation is in range $100\,\text{Å}$ to $150\,\text{Å}$ $\Rightarrow \boxed{Z = 3}$ (a)

The highest energy photon has energy $hf = \frac{hc}{\lambda}$ with $\lambda = 100\,\text{Å} \Rightarrow$

$\boxed{hf = 124\,eV}$. The ionization energy is $E_0 Z^2 = 13.6\,eV \times 9 = \boxed{122.4\,eV}$

$\Rightarrow$ maximum kinetic energy of electron $= 124\,eV - 122.4\,eV = \boxed{1.6\,eV}$ (b)

(c) Wavelength required for transitions from $n=1$ to $m>1$:

$$\frac{hc}{\lambda} = Z^2 E_0 \cdot \left(1 - \frac{1}{m^2}\right) \Rightarrow \lambda_{1m} = \frac{hc}{Z^2 E_0} \frac{1}{1 - \frac{1}{m^2}} = \frac{101.307\,\text{Å}}{1 - \frac{1}{m^2}}$$

$$\Rightarrow \lambda_{12} = \frac{101.307\,\text{Å}}{\frac{3}{4}} = 135.08\,\text{Å}$$

$$\lambda_{13} = \frac{101.307\,\text{Å}}{\frac{8}{9}} = 113.97\,\text{Å}$$

2 absorption lines are seen if incident wavelengths are in range $110\,\text{Å}$ to $150\,\text{Å}$ (c)

$$\lambda_{14} = \frac{101.307\,\text{Å}}{\frac{15}{16}} = 108.06\,\text{Å}$$

$\lambda_{1m} < \lambda_{14}$ for $m > 4$

Problem 4

$$E_1 = \frac{\hbar^2}{2m_e}\frac{\pi^2}{L^2} = \frac{3.81\,eV\text{Å}^2 \cdot \pi^2}{25\text{Å}^2} = \boxed{1.50\,eV} \quad (a)$$

(b) Inside the barrier, the electron wavefunction is

$$\psi \sim e^{-\alpha x} = e^{-x/\delta}$$

with $\alpha = \sqrt{\frac{2m_e}{\hbar^2}(U-E)} = \sqrt{\frac{1}{3.81\,eV\text{Å}^2}(50eV - 1.5eV)} = \frac{3.57}{\text{Å}}$

$\Rightarrow \delta = 1/\alpha = 0.28\,\text{Å}$

So the effective width of the well is $\sim 5\text{Å} + \delta = 5.28\,\text{Å}$.

So a better estimate of the ground state energy is

$$E_1' = \frac{\hbar^2\pi^2}{2m_e(L+\delta)^2} = 1.50\,eV \cdot \frac{L^2}{(L+\delta)^2} = 1.50\,eV\left(\frac{5}{5.28}\right)^2 = 1.35\,eV$$

$\Rightarrow \boxed{E_1' = 1.35\,eV}$

(c)

$$T = e^{-2\alpha L} = e^{-2\times 3.57\text{Å}^{-1}\times 1\text{Å}} = e^{-7.14} = 7.9\times 10^{-4}$$

(d) Momentum $p = m_e v = \hbar k = \frac{\hbar\pi}{L} \Rightarrow v = \frac{\hbar\pi}{m_e L} \Rightarrow$

$$\frac{v}{c} = \frac{\hbar c\,\pi}{m_e c^2 L} = \frac{1973\,eV\text{Å}\cdot\pi}{0.511\cdot 10^6 eV\, 5\text{Å}} = \boxed{0.0024} \Rightarrow \boxed{v = 7.3\times 10^{+15}\frac{\text{Å}}{s}}$$

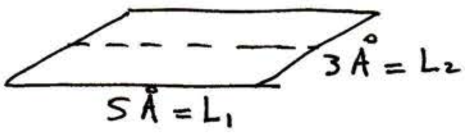
(e) The electron travels $2L = 10\text{Å}$ every time it hits the right wall.

So in 1s it hits $\frac{v}{2L}$ times $= \boxed{7.3\times 10^{14}\text{ times}}$

(f) If every time it hits barrier the probability of tunneling out is $T = 7.9\times 10^{-4}$

the probability it will tunnel out in a 1s time interval is ~ 1

Problem 5



(a) $E_{n_1,n_2} = \dfrac{\hbar^2\pi^2}{2m_e}\left(\dfrac{n_1^2}{L_1^2} + \dfrac{n_2^2}{L_2^2}\right) = \dfrac{\hbar^2\pi^2}{2m_e L_1^2}\left(n_1^2 + n_2^2\dfrac{L_1^2}{L_2^2}\right) =$

$= 1.50\,eV\left(n_1^2 + n_2^2 \times 2.778\right)$

$\Rightarrow \; E_{11} = 5.67eV, \quad E_{21} = 10.17eV, \quad E_{12} = 18.17eV, \quad E_{31} = 17.66eV$

So the 3 lowest levels are:

$n_1, n_2 = 1, 1 \quad ; \quad E_{11} = 5.76\,eV$

$n_1, n_2 = 2, 1 \quad ; \quad E_{21} = 10.17eV$

$n_1, n_2 = 3, 1 \quad ; \quad E_{31} = 17.66eV$

(b) To have $\Psi(x, y = L_2/2) = 0$ we need $n_2 = 2$, so lowest state is

$\boxed{(n_1, n_2) = (1,2)}$; $\boxed{E_{12} = 18.17\,eV.}$ Wavefunction is

$$\boxed{\Psi(x,y) = \sqrt{\frac{2}{L_1}}\sqrt{\frac{2}{L_2}}\,\sin\frac{\pi x}{L_1}\,\sin\frac{2\pi y}{L_2}}$$

(c) For 5 electrons:

$E = 2E_{11} + 2E_{21} + E_{31} = 49.34\,eV$

the lowest excitation is electron in $31 \to 12$ $\Rightarrow$

$\Delta E = 18.17eV - 17.66eV = 0.51eV = \dfrac{hc}{\lambda}$

$$\Rightarrow \boxed{\lambda = \frac{hc}{\Delta E} = 24{,}314\,\mathring{A}}$$

Problem 6

$$\psi(r,\theta,\phi) = C r^2 e^{-r/3a_0} \sin^2\theta\, e^{i\phi}$$

(a) since exponential part is e^{-r/na_0} $\Rightarrow \boxed{n=3}$

Since $R(r)$ has no nodes, $l = n-1 \Rightarrow \boxed{l=2}$

Since azimuthal part is $e^{im_l\phi} \Rightarrow \boxed{m_l = 1}$

(b) $P(r) = r^2 R^2(r) = C^2 r^6 e^{-2r/3a_0}$

$P'(r) \propto 6r^5 - \frac{2}{3a_0} r^6 = 0 \Rightarrow r = \frac{6 \cdot 3a_0}{2} \Rightarrow \boxed{r = 9a_0}$

In the Bohr atom, $r_n = n^2 a_0$, $n = 3 \Rightarrow \boxed{r_3 = 9a_0 \text{ same}}$

(c) Normalization:

$$1 = \int_0^\infty dr\, P(r) = C^2 \int_0^\infty dr\, r^6 e^{-2r/3a_0} \Rightarrow \boxed{C^2 = \frac{1}{\int_0^\infty dr\, r^6 e^{-2r/3a_0}}}$$

$$\langle r\rangle = \int_0^\infty dr\, r P(r) = C^2 \int_0^\infty dr\, r^7 e^{-2r/3a_0} = \frac{\int_0^\infty dr\, r^7 e^{-2r/3a_0}}{\int_0^\infty dr\, r^6 e^{-2r/3a_0}}$$

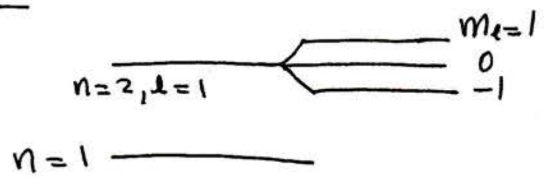
Using $\int_0^\infty dr\, r^n e^{-\lambda r} = \frac{n!}{\lambda^{n+1}} \Rightarrow$

$$\langle r\rangle = \frac{7!}{\left(\frac{2}{3a_0}\right)^8} \cdot \frac{\left(\frac{2}{3a_0}\right)^7}{6!} = \frac{7}{\frac{2}{3a_0}} = \frac{21}{2} a_0 = \boxed{10.5 a_0}$$

(d) $$\left\langle \frac{1}{r}\right\rangle = \int_0^\infty dr\, \frac{1}{r} P(r) = \frac{5!}{\left(\frac{2}{3a_0}\right)^6} \cdot \frac{\left(\frac{2}{3a_0}\right)^7}{6!} = \frac{1}{6} \cdot \frac{2}{3a_0} = \boxed{\frac{1}{9a_0}}$$

In the Bohr atom, $\boxed{\frac{1}{r_n} = \frac{1}{n^2 a_0} = \frac{1}{9a_0} \Rightarrow \text{same}}$

Problem 7



in magnetic field: $U = -\vec{\mu}\cdot\vec{B} = -\mu_z B$

without spin, $\mu_z = -\mu_B m_\ell \Rightarrow U = +\mu_B B m_\ell$

So: $E_2 \rightarrow E_2 + \mu_B B m_\ell$

$$\mu_B B m_\ell = 5.79\times10^{-5}\,eV\cdot 15\, m_\ell = \boxed{8.69\times10^{-4}\, m_\ell\cdot eV}$$

So energies of photons emitted in the transition from $n=2$ to $n=1$ are

$$\boxed{hf = E_2 - E_1 \pm 8.69\times10^{-4} \quad (m_\ell=\pm1) \text{ and } E_2-E_1 \ (m_\ell=0)}$$

with $E_2 - E_1 = 13.6\,eV \times \frac{3}{4} = 10.2\,eV$

(b) With spin: $\mu_z = -\frac{e}{2m_e}(\hbar m_\ell + 2\hbar m_s) = -\mu_B(m_\ell + 2m_s)$

$$U = -\mu_z B = \mu_B B(m_\ell + 2m_s) = \mu_B B \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \begin{array}{l} m_\ell=1, m_s=\frac{1}{2} \\ m_\ell=0, m_s=\frac{1}{2} \\ m_\ell=1, m_s=-\frac{1}{2} \,/\, m_\ell=-1, m_s=\frac{1}{2} \\ m_\ell=0, m_s=-\frac{1}{2} \\ m_\ell=-1, m_s=-\frac{1}{2} \end{array}$$

(c) The difference in energy between the $m_\ell=-1$ and $m_\ell=0$ states is $8.69\times10^{-4}\,eV$

At low temperatures, most electrons will be in lowest energy state $m_\ell=-1$.

Since the relative probability is given by the Boltzmann factor

$e^{-(E(m_\ell=0)-E(m_\ell=-1))/k_BT}$, this becomes small when

$$k_BT < E(m_\ell=0) - E(m_\ell=-1) \Rightarrow \boxed{T < \frac{8.69\times10^{-4}\,eV}{\frac{1}{11{,}600}\,\frac{eV}{°K}} = 10.1\,°K}$$

Problem 8

$\Psi(x) = C\cos(kx)$, $[p] = \frac{\hbar}{i}\frac{\partial}{\partial x}$

(a) Q is a sharp observable if $[Q]\Psi = q\Psi$, with q a number.

$[p]\Psi = \frac{\hbar}{i}\frac{\partial}{\partial x} C\cos(kx) = -\frac{\hbar k}{i} C\sin(kx) \neq q \cdot C\cos(kx)$

$\Rightarrow$ $\boxed{\text{p \underline{is not} a sharp observable}}$

(b) $[p^2] = \frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x} = -\hbar^2\frac{\partial^2}{\partial x^2}$

$[p^2]\Psi = -\hbar^2\frac{\partial^2}{\partial x^2} C\cos(kx) = \hbar^2 k^2 C\cos(kx) = q\Psi$

with $q = \hbar^2 k^2$ a number. So $\boxed{p^2 \text{ \underline{is} a sharp observable}}$

(c) Since p^2 is sharp, there is no uncertainty, $\boxed{\Delta(p^2) = 0.}$

$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$

$\langle p\rangle = 0$ because wave function is even: $\int_{-a}^{a} dx \cos(kx)\frac{\hbar}{i}\frac{\partial}{\partial x}\cos(kx) = 0$

$\langle p^2\rangle = \hbar^2 k^2 \int dx\, |\Psi(x)|^2 = \hbar^2 k^2 \cdot 1$ since Ψ is normalized

$\Rightarrow$ $\boxed{\Delta p = \hbar k}$