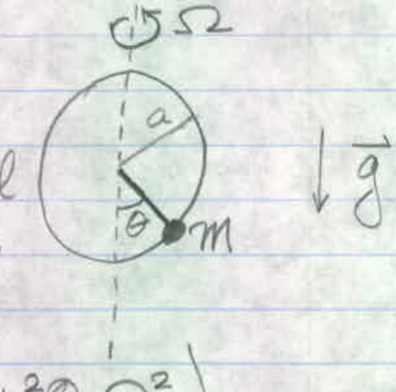
Luke Johns
Phys 200A

Homework #1

(1) a) There are two sources of KE: motion along the wire hoop and rotational motion due to $\vec{\Omega}$. Thus



$$T = \frac{1}{2} m \left(a^2 \dot{\theta}^2 + a^2 \sin^2\theta \, \Omega^2 \right).$$

The PE is gravitational:

$$V = mga(1-\cos\theta).$$

Hence $$\boxed{L = \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \sin^2\theta \, \Omega^2 \right) - mga(1-\cos\theta).}$$

b) An equilibrium circular orbit satisfies

$$\dot{\theta} = \ddot{\theta} = 0.$$

We impose this condition on the EOM:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left[m a^2 \dot{\theta} \right] = m a^2 \ddot{\theta}$$

$$= \frac{\partial L}{\partial \theta} = m a^2 \sin\theta \cos\theta \, \Omega^2 - mga \sin\theta$$

$$\Rightarrow \ddot{\theta} = 0 = \sin\theta \cos\theta \, \Omega^2 - \frac{g}{a} \sin\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{g}{a\Omega^2}.}$$

Now we switch to a co-rotating coordinate system. In this reference frame, the mass in circular orbit appears stationary. We have

$$\vec{F}_{phys} - \vec{F}_{centrif} = m\frac{d^2\vec{r}}{dt^2} = 0$$

$$\Longrightarrow \vec{F}_g + \vec{F}_N - \vec{F}_{centrif} = 0$$

$$\Longrightarrow -mg\hat{z} + \overbrace{m\Omega^2 a\left(\cos\theta\,\hat{z} - \sin\theta\,\hat{y}\right)}^{\text{since } \vec{F}_N \text{ must point toward the hoop's center}} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = 0.$$

The centrifugal force is

$$m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = m\Omega^2\,\hat{z}\times\left(\hat{z}\times\left(a\sin\theta\,\hat{y} - a\cos\theta\,\hat{z}\right)\right)$$

$$= -m\Omega^2 a\sin\theta\,\hat{y}.$$

therefore the $\hat{z}$-equation reads

$$mg = m\Omega^2 a\cos\theta$$

$$\Longrightarrow \boxed{\cos\theta = \frac{g}{a\Omega^2}.}$$

c) Let $\theta = \theta_0 + \eta$, where η is small and θ_0 satisfies

$$\cos\theta_0 = \frac{g}{a\Omega^2}.$$

Plugging into the EOM yields

$$\ddot{\theta} = \ddot{\eta} = \sin(\theta_0+\eta)\cos(\theta_0+\eta)\,\Omega^2 - \frac{g}{a}\sin(\theta_0+\eta)$$

$$= \sin\theta_0\cos\theta_0\,\Omega^2 - \frac{g}{a}\sin\theta_0 + \eta\left[(\cos^2\theta_0 - \sin^2\theta_0)\Omega^2 - \frac{g}{a}\cos\theta_0\right]$$

$$= -\eta\left[\Omega^2\cos^2\theta_0 + \Omega^2(\sin^2\theta_0 - \cos^2\theta_0)\right]$$

$$\Longrightarrow \boxed{\ddot{\eta} = -\underbrace{\Omega^2\sin^2\theta_0}_{\omega^2}\,\eta.}$$

d) If $a\Omega^2 < g$, then an equilibrium circular orbit would require

$$\cos\theta = \frac{g}{a\Omega^2} > 1.$$

Clearly this cannot be satisfied, so no such orbit exists.

Phys 200A (Theoretical Mechanics), Problem Set I

Fetter & Walecka, problem #3.2.

Done by Munirov V. R.

1) Lagrangian

The Lagrangian of the system is straightforward:

$L = T - V,$

$L = \frac{m}{2}\dot{l}^2 + \frac{m}{2}\Omega^2 l^2 \sin^2\theta_0 - mgl\cos\theta_0.$

2) Equilibrium orbit

Euler-Lagrange equations of motion:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) - \frac{\partial L}{\partial l} = 0,$

$m\ddot{l} = m\Omega^2 \sin^2\theta_0 l - mg\cos\theta_0.$

In equilibrium $\ddot{l} = 0$, so we get the condition for an equilibrium circular orbit:

$l_0 = \frac{g\cos\theta_0}{(\Omega\sin\theta_0)^2}$, QED.

3) Stability

To consider the stability of this orbit against small displacements along the wire we write $l = l_0 + \triangle l$ ($\Delta l \to 0$) and put it into the equation of motion. By doing that we get:

$\Delta\ddot{l} = \Omega^2 \sin^2\theta_0 \triangle l,$

which tells us that it is **unstable equilibrium** because coefficient before $\triangle l$ is positive.

4) Balance of force

In non-inertial rotational reference frame there are three forces acting on the point mass: gravitational force $m\mathbf{g}$ acting downward, centrifugal force $m\Omega^2 l\sin\theta_0$ acting outward from rotational orbit and reaction force of the wire $m\Omega^2\sin\theta_0\cos\theta_0 l + mg\sin\theta_0$ pointing perpendicular to it . These three forces balance each other.

It is possible to obtain the expression for the reaction force using the method of Lagrange multipliers.

$L = \frac{m}{2}\dot{l}^2 + \frac{m}{2}l^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 l^2\sin^2\theta_0 - mgl\cos\theta_0,$

$f(\theta) = \theta = \theta_0$ - constraints.

Equation of motion with constraints:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda\frac{f}{\partial\theta}$, where λ - Lagrange multiplier.

$ml^2\ddot{\theta} = m\Omega^2 l^2\cos\theta\sin\theta + mgl\sin\theta + \lambda,$

$\ddot{\theta} = 0,\ \theta = \theta_0.$

$N = \frac{\lambda}{l} = -m\Omega^2 l\cos\theta_0\sin\theta_0 - mg\sin\theta_0.$

① Elizabeth Wicks

$\boxed{3.7}$ (a)

$L = T - V$

$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + 2l\dot{\theta}\cos\theta\,\dot{x} + l^2\dot{\theta}^2)$

$V_g = -mgl\cos\theta$

$V_{springs} = \frac{1}{2}\frac{k}{2}x^2 + \frac{1}{2}\frac{k}{2}x^2 = \frac{1}{2}kx^2$

$\Rightarrow \boxed{L = \frac{1}{2}m(\dot{x}^2 + 2l\dot{\theta}\cos\theta\,\dot{x} + l^2\dot{\theta}^2) + mgl\cos\theta - \frac{1}{2}kx^2}$

calculate the equations of motion.

EOM_x: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

$\frac{d}{dt}(m\dot{x} + ml\dot{\theta}\cos\theta) - (-kx) = 0$

$m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}\sin\theta\,\dot{\theta} + kx = 0$

$\boxed{m\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta(\dot{\theta})^2 + kx = 0}$

EOM_y: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

$0 = \frac{d}{dt}(ml\cos\theta\,\dot{x} + ml^2\dot{\theta}) - (-ml\dot{\theta}\sin\theta\,\dot{x} - mgl\sin\theta)$

$= \cancel{-ml\sin\theta\,\dot{x}\,\dot{\theta}} + ml\cos\theta\,\ddot{x} + \cancel{ml\dot{\theta}\sin\theta\,\dot{x}} + mgl\sin\theta$
$+ ml^2\ddot{\theta}$

$\Rightarrow \boxed{\cos\theta\,\ddot{x} + g\sin\theta + l\ddot{\theta} = 0}$

(2)

(b) For small θ...

Our EOM transform to:

θ: $\ddot{x} + g\theta + l\ddot{\theta} = 0$

$$\Rightarrow \boxed{\frac{\ddot{x}}{l} + \ddot{\theta} + \frac{g}{l}\theta = 0.}$$

x: $m\ddot{x} + kx + ml\ddot{\theta} - ml\theta(\dot{\theta})^2 = 0$

$\ddot{x} + \frac{k}{m}x + l\ddot{\theta} - l\theta(\dot{\theta})^2 = 0$

ignore non-linear terms

$$\boxed{\frac{\ddot{x}}{l} + \frac{k}{m}\left(\frac{x}{l}\right) + \ddot{\theta} = 0}$$

Set new variables $z = \frac{x}{l}$, $\omega_g = \sqrt{\frac{g}{l}}$, $\omega_k = \sqrt{\frac{k}{m}}$.

$$\ddot{z} + \ddot{\theta} + \omega_g^2\theta = 0$$

$$\ddot{z} + \omega_k^2 z + \ddot{\theta} = 0$$

Assume periodic solutions for some ω,

$\theta = ae^{i\omega t}$ $\quad z = be^{i\omega t}$.

Then...

$-b\omega^2 e^{i\omega t} - a\omega^2 e^{i\omega t} + \omega_g^2 a e^{i\omega t} = 0$

$\Rightarrow -\omega^2 z - \omega^2\theta + \omega_g^2\theta = 0$

$\Rightarrow -\omega^2 z + (\omega_g^2 - \omega^2)\theta = 0$

and

$-b\omega^2 e^{i\omega t} + \omega_k^2 z - a\omega^2 e^{i\omega t} = 0$

$\Rightarrow (\omega_k^2 - \omega^2) z - \omega^2\theta = 0$

$$\Rightarrow \begin{pmatrix} -\omega^2 & \omega_g^2 - \omega^2 \\ \omega_k^2 - \omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} z \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ for all z, θ

(3)

(b) continued

$$\Rightarrow \det\begin{pmatrix} -\omega^2 & \omega_g^2-\omega^2 \\ \omega_k^2-\omega^2 & -\omega^2 \end{pmatrix} = 0$$

$$\Rightarrow \omega^4 - (\omega_g^2-\omega^2)(\omega_k^2-\omega^2) = 0$$

$$\omega^4 - \left((\omega_g\omega_k)^2 - \omega^2(\omega_g^2+\omega_k^2) + \omega^4\right) = 0$$

$$\Rightarrow -(\omega_g\omega_k)^2 + \omega^2(\omega_g^2+\omega_k^2) = 0$$

$$\Rightarrow \quad \omega^2 = \frac{\omega_g^2\omega_k^2}{\omega_g^2+\omega_k^2} = \frac{\frac{g}{l}\frac{k}{m}}{\frac{g}{l}+\frac{k}{m}}$$

$$= \frac{gk}{lm\left(\frac{g}{l}+\frac{k}{m}\right)} = \frac{gk}{mg+lk}$$

$$= \frac{g}{\frac{mg}{k}+l} \Rightarrow \boxed{\omega = \sqrt{\frac{g}{\frac{mg}{k}+l}}}$$

This is the same frequency of oscillation as a pendulum with length $\frac{mg}{k}+l$.

(4)

Velocity calculation:

$$\vec{\dot{x}} + \vec{v}' = \vec{v}$$

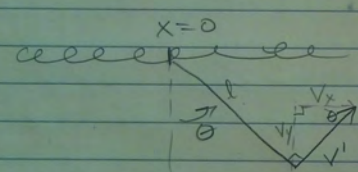
$$v' = l\dot{\theta} \qquad \Rightarrow v_x = \dot{x} + l\dot{\theta}\cos\theta$$

$$v'_x = l\dot{\theta}\cos\theta \qquad v_y = l\dot{\theta}\sin\theta$$

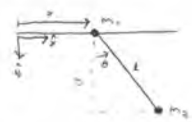
$$v'_y = l\dot{\theta}\sin\theta$$

$$\Rightarrow v^2 = v_x^2 + v_y^2 = (\dot{x})^2 + l^2\dot{\theta}^2\cos^2\theta + 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2\sin^2\theta$$

$$= \boxed{(\dot{x})^2 + 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2}$$



$\boxed{3}$

3. 

Select generalized coordinates x and θ

i. Energy contributions,

$$T_1 = \frac{1}{2} m_1 \dot{x}^2 \qquad T_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right), \quad x_2 = x + l\sin\theta$$

$$V_1 = 0 \qquad y_2 = l\cos\theta$$

$$\dot{x}_2 = \dot{x} + l\cos\theta\,\dot{\theta}$$

$$\dot{y}_2 = -l\sin\theta\,\dot{\theta}$$

$$T_2 = \frac{1}{2} m_2 \left[\dot{x}^2 + 2\dot{x} l \dot{\theta} \cos\theta + l^2 \cos^2\theta\,\dot{\theta}^2 + l^2 \sin^2\theta\,\dot{\theta}^2 \right]$$

$$V_2 = m_2 g l \left(1 - \cos\theta \right)$$

ii. Lagrangian,

$$L = \sum_{i=1}^{2} (T_i - V_i) = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 \dot{x} l \dot{\theta} \cos\theta + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 g l \cos\theta - m_2 g l$$

iii. Euler-Lagrange eqns of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left[m_2 \dot{x} l \cos\theta + m_2 l^2 \dot{\theta} \right] - \left[-m_2 \dot{x} l \dot{\theta} \sin\theta - m_2 g l \sin\theta \right] = 0$$

$$m_2 \ddot{x} l \cos\theta + \cancel{m_2 \dot{x} l \sin\theta\,\dot{\theta}} + m_2 l^2 \ddot{\theta} + \cancel{m_2 \dot{x} l \dot{\theta}} \sin\theta + m_2 g l \sin\theta = 0$$

$$\ddot{\theta}(m_2 l^2) + m_2 \ddot{x} l \cos\theta + m_2 g l \sin\theta = 0$$

assume θ is small,

$$\ddot{\theta}(\cancel{m_2} l^2) + \cancel{m_2} \ddot{x} \cancel{l} + \cancel{m_2} g l \theta = 0$$

(1) $$\ddot{\theta} = -\frac{g}{l}\theta - \frac{\ddot{x}}{l}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left[(m_1 + m_2)\dot{x} + m_2 l \dot{\theta} \cos\theta \right] - 0 = 0$$

$$(m_1 + m_2)\ddot{x} + m_2 l \ddot{\theta} \cos\theta + -m_2 l \dot{\theta} \sin\theta\,\dot{\theta} = 0$$

$$\ddot{x} = \frac{m_2 l \left(\sin\theta\,\dot{\theta}^2 - \cos\theta\,\ddot{\theta} \right)}{m_1 + m_2}$$

(2) $$\ddot{x} = \frac{m_2 l \left(\theta \dot{\theta}^2 - \ddot{\theta} \right)}{m_1 + m_2}$$

iii. Combine Eqs (1) + (2) to find θ-differential equation,

$$\ddot{\theta} = -(g/\ell)\theta - (1/\ell)\left[\frac{m_2\ell(\theta\dot{\theta}^2 - \ddot{\theta})}{m_1+m_2}\right]$$

$$\ddot{\theta}\left(1 - m_2/m_1+m_2\right) + \frac{m_2}{m_1+m_2}\theta\dot{\theta}^2 + (g/\ell)\theta = 0$$

and $\dot{\theta}^2 \sim 0$,

$$\ddot{\theta} = -\frac{g/\ell}{(1 - m_2/m_1+m_2)}\theta$$

$$\boxed{\omega^2 = \frac{g/\ell}{(1 - m_2/m_1+m_2)}}$$

3.8

A point mass m slides without friciton inside a surface of revolution $z = \alpha \sin(r/R)$ whose symmetry axis lied along the direction of a uniform gravitational field $\mathbf{g}$. Consider $0 < r/R < \frac{1}{2}\pi$.

(a) Construct the lagrangian L and compute the equations of motion for the generalized coordinates r and ϕ. The lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \left(\frac{\alpha}{R}\right)^2 \dot{r}^2\cos^2\left(\frac{r}{R}\right)\right) - mg\alpha\sin\left(\frac{r}{R}\right).$$

The equation of motion are

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0,$$

$$m\ddot{r}\left(1 + \left(\frac{\alpha}{R}\right)^2\cos^2\left(\frac{r}{R}\right)\right) = mr\dot{\phi}^2 + m\left(\frac{\alpha}{R}\right)^2\dot{r}^2\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) - mg\frac{\alpha}{r}\cos\left(\frac{r}{R}\right).$$

Note that angular momentum is conserved in this system, $L \equiv mr^2\dot{\phi}$, so the equation of motion for generalized coordinate r can be rewritten as

$$m\ddot{r}\left(1 + \left(\frac{\alpha}{R}\right)^2\cos^2\left(\frac{r}{R}\right)\right) = \frac{L^2}{mr^3} + m\left(\frac{\alpha}{R}\right)^2\dot{r}^2\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) - mg\frac{\alpha}{R}\cos\left(\frac{r}{R}\right).$$

(b) Are there stationary horizontal circular orbits? This problem can be solved using the equation of motion or the effective potential, $U_{\text{eff}} = U + L^2/(2mr^2)$. A stationary point, r_0 is defined by,

$$\left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r=r_0} = 0.$$

For the effective potential at hand

$$\frac{\partial U_{\text{eff}}}{\partial r} = -\frac{L^2}{mr^3} + mg\frac{\alpha}{R}\cos\left(\frac{r}{R}\right),$$

which leads to the transcendental equation

$$\frac{L^2}{m^2R^2g\alpha} = \tilde{r}_0^3\cos(\tilde{r}_0),$$

where $\tilde{r}_0 = r_0/R$. Depending on the value left-hand side of the above equation there can be 0, 1, or 2 stationary points.

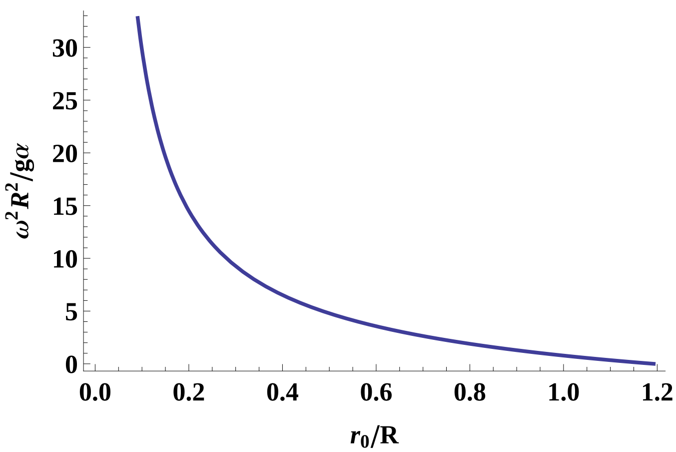


Figure 1: The frequency of oscillation about the equilibrium orbit.

(c) Which of these orbits is stable under small impluses along the surface transverse to the direction of motion? An orbit is stable if the second derivative of the effective potential is positive,

$$\frac{\partial^2 U_{\rm eff}}{\partial r^2} = \frac{3L^2}{mr^4} - mg\frac{\alpha}{R^2}\sin\left(\frac{r}{R}\right).$$

This leads to the condition that a stationary point is stable if

$$\frac{3L^2}{m^2R^2g\alpha} > \tilde{r}_0^4\sin(\tilde{r}_0).$$

Substituting our result from the stationary point analysis into this equation yields

$$3 > \tilde{r}_0\tan(\tilde{r}_0).$$

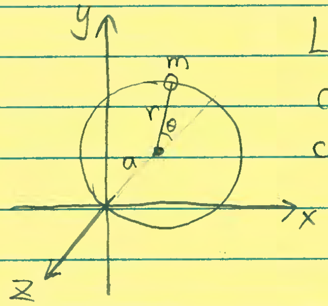
Solving this numerically, we see that a stationary point is stable if $\tilde{r}_0 < 1.19$.

(d) If the orbit is stable, what is the frequency of oscillation about the equilibrium orbit? The frequency ω is given by

$$m\omega^2 = \left.\frac{\partial^2 U_{\rm eff}}{\partial r^2}\right|_{r=r_0}.$$

Because there are transcendental equations in the problem it will be easier to proceed numerically. Figure 1 shows $\omega^2R^2/g\alpha$ as a function of r_0/R in the region where the stationary point is stable.

3.15a



Let r and θ be the generalized coordinates. x and y be the Cartesian coordinates of the bead.

$$\begin{cases} x = a\cos\omega t + r\cos(\omega t+\theta) \\ y = a\sin\omega t + r\sin(\omega t+\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = -a\omega\sin\omega t + \dot{r}\cos(\omega t+\theta) - r(\omega+\dot{\theta})\sin(\omega t+\theta) \\ \dot{y} = a\omega\cos\omega t + \dot{r}\sin(\omega t+\theta) + r(\omega+\dot{\theta})\cos(\omega t+\theta) \end{cases}$$

In this case $L = T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$

$$\frac{2}{m}L = a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t + \dot{r}^2\cos^2(\omega t+\theta) + \dot{r}^2\sin^2(\omega t$$
$$+ r^2(\omega+\dot{\theta})^2\sin^2(\omega t+\theta) + r^2(\omega+\dot{\theta})^2\cos^2(\omega t+\theta) -$$
$$2a\omega\dot{r}\sin\omega t\cos(\omega t+\theta) + 2a\omega r(\omega+\dot{\theta})\sin\omega t\sin(\omega t+\theta$$
$$- 2r\dot{r}(\omega+\dot{\theta})\cos(\omega t+\theta)\sin(\omega t+\theta) + 2a\omega\dot{r}\cos\omega t\sin(\omega t$$
$$+ 2a\omega r(\omega+\dot{\theta})\cos\omega t\cos(\omega t+\theta) + 2r\dot{r}(\omega+\dot{\theta})\sin(\omega t+\theta)\cos(\omega t$$

$$L = \left(a^2\omega^2 + \dot{r}^2 + r^2(\omega+\dot{\theta})^2 + 2a\omega\dot{r}\sin\theta + 2a\omega r(\omega+\dot{\theta})\cos\theta\right)\frac{m}{2}$$

$$\left.\begin{cases} ① \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda\frac{\partial f}{\partial r} \\ ② \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f}{\partial \theta} \\ ③ \ f = r - a = 0 \end{cases}\right\}$$ 3 equations, 3 unknowns r, θ, λ.

$$\Rightarrow \begin{cases} ① \ \left[\frac{d}{dt}(2\dot{r} + 2a\omega\sin\theta) - 2r(\omega+\dot{\theta})^2 - 2a\omega(\omega+\dot{\theta})\cos\theta\right]\frac{m}{2} = \lambda \\ \left[2\ddot{r} + \cancel{2a\omega\dot{\theta}\cos\theta} - 2r(\omega+\dot{\theta})^2 - 2a\omega(\omega+\cancel{\dot{\theta}})\cos\theta\right]\frac{m}{2} = \lambda \\ \left[2\ddot{r} - 2r(\omega+\dot{\theta})^2 - 2a\omega^2\cos\theta\right]\frac{m}{2} = \lambda \\ ② \ \frac{d}{dt}\left(2r^2(\omega+\dot{\theta}) + 2a\omega r\cos\theta\right) - 2a\omega\dot{r}\cos\theta + 2a\omega r(\omega+\dot{\theta})\sin = 0 \\ 4r\dot{r}(\omega+\dot{\theta}) + 2r^2\ddot{\theta} + \cancel{2a\omega\dot{r}\cos\theta} - \cancel{2a\omega r\dot{\theta}\sin\theta} \\ \qquad - \cancel{2a\omega\dot{r}\cos\theta} + 2a\omega r(\omega+\cancel{\dot{\theta}})\sin\theta = \\ 4r\dot{r}(\omega+\dot{\theta}) + 2r^2\ddot{\theta} + 2a\omega^2 r\sin\theta = 0 \\ ③ \ r = a \qquad (\dot{r} = \ddot{r} = 0) \end{cases}$$

plug ③ into ① and ②.