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PHYS 200A. Theoretical Mechanics.

Homework 5:

4.1: (a). ① Hoop: Moment of inertia for hoop:

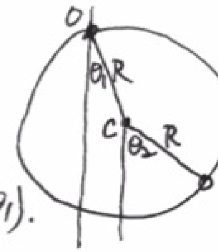
$$I_c = MR^2.$$

Then use Parallel Axis Theorem to change the reference point:

$$I = I_c + MR^2 = 2MR^2.$$

$$\therefore T_h = \frac{1}{2} I \dot{\theta}_1^2 = MR^2 \dot{\theta}_1^2$$

$$V_h = -MgR\cos\theta_1 = -MgR + \frac{1}{2} MgR\theta_1^2.$$



② Bead: $\vec{v}$ is $\vec{v}_c$ + velocity relative to C.

$$\therefore v^2 = (R\dot{\theta}_1)^2 + (R\dot{\theta}_2)^2 + 2(R\dot{\theta}_1)(R\dot{\theta}_2)\cos(\theta_2 - \theta_1).$$

$$= R^2\dot{\theta}_1^2 + R^2\dot{\theta}_2^2 + 2R^2\dot{\theta}_1\dot{\theta}_2.$$

$$\therefore T_b = \frac{1}{2}Mv^2 = \frac{1}{2}MR^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2)$$

$$V_b = -MgR\cos\theta_1 - MgR\cos\theta_2 = -2MgR + \frac{1}{2}MgR\theta_1^2 + \frac{1}{2}MgR\theta_2^2.$$

$$\therefore L = T_b + T_h - V_h - V_b = \frac{1}{2}MR^2(3\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) - \frac{1}{2}MgR(2\theta_1^2 + \theta_2^2) + 3MgR.$$

Compare to the standard form:

$$L = \frac{1}{2}\sum_\alpha \sum_\lambda (m_{\lambda\sigma}\dot{\theta}_\lambda\dot{\theta}_\sigma - V_{\lambda\sigma}\theta_\lambda\theta_\sigma) - V_0.$$

$$\therefore m_{ij} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} MR^2.$$

$$V_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} MgR. \qquad \left(\text{Let } \theta_i = C\rho_i \cdot e^{i\omega t}\right)$$

The condition to have nontrivial solution is $\det(V_{\sigma\lambda} - \omega^2 m_{\sigma\lambda}) = 0$.

i.e.

$$\det\begin{pmatrix} 2MgR - \omega^2 \cdot 3MR^2 & -\omega^2 MR^2 \\ -\omega^2 MR^2 & MgR - \omega^2 MR^2 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{cases} \omega_1 = \frac{1}{2}\sqrt{\frac{2g}{R}} \\ \omega_2 = \sqrt{\frac{2g}{R}} \end{cases}.$$

(b). Plug in ω_1, ω_2 into the following equ respectively:

$$\begin{pmatrix} 2MgR - \omega^2 \cdot 3MR^2 & -\omega^2 MR^2 \\ -\omega^2 MR^2 & MgR - \omega^2 MR^2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = 0.$$

Plug in ω_1 and we get: $\rho_2^{(1)} = \rho_1^{(1)}$.

Plug in ω_2 and we get: $\rho_2^{(2)} = -2\rho_1^{(2)}$.
Now normalize it, according to $\sum_\lambda \sum_\sigma \rho_\sigma^{T(t)} m_{\sigma\lambda} \rho_\lambda^{(s)} = \delta_{st}$.

$\therefore \quad \rho^{(1)} = \frac{1}{\sqrt{6MR^2}}\begin{pmatrix}1\\1\end{pmatrix}. \qquad \rho^{(2)} = \frac{1}{\sqrt{3MR^2}}\begin{pmatrix}1\\-2\end{pmatrix} = \frac{1}{\sqrt{6MR^2}}\begin{pmatrix}\sqrt{2}\\-2\sqrt{2}\end{pmatrix}.$

(c). $A = \frac{1}{\sqrt{6MR^2}}\begin{pmatrix}1 & \sqrt{2}\\ 1 & -2\sqrt{2}\end{pmatrix}.$

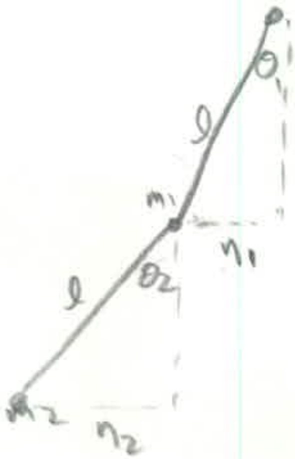
(d). $\zeta = A^T m \theta = \begin{pmatrix}1 & 1\\ \sqrt{2} & -2\sqrt{2}\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\cdot\frac{1}{\sqrt{6MR^2}}\cdot MR^2 = \sqrt{\frac{MR^2}{6}}\cdot\begin{pmatrix}4\theta_1+2\theta_2\\ \sqrt{2}\theta_1-\sqrt{2}\theta_2\end{pmatrix}.$

Explicitly, $\zeta_1 = \sqrt{\frac{MR^2}{6}}(4\theta_1+2\theta_2)$, $\quad \zeta_2 = \sqrt{\frac{MR^2}{6}}(\sqrt{2}\theta_1 - \sqrt{2}\theta_2)$.

It's easy to show $L = \frac{1}{2}\dot{\zeta}_1^2 + \frac{1}{2}\dot{\zeta}_2^2 - \frac{1}{2}\omega_1^2\zeta_1^2 - \frac{1}{2}\omega_2^2\zeta_2^2$ is equivalent to the original L.

4.3

a)



$$\theta_i \approx \sin\theta_i = \frac{\eta_i}{l}$$

$$x_1 = \eta_1$$

$$x_2 = \eta_1 + \eta_2$$

$$y_1 = -l\cos\theta_1 \approx -l + \frac{l}{2}\,\theta_1^2 = -l + \frac{l}{2}\frac{\eta_1^2}{l^2} = -l + \frac{\eta^2}{2l}$$

$$y_2 = -l\cos\theta_1 - l\cos\theta_2 \approx -2l + \frac{l}{2}\left(\theta_1^2 + \theta_2^2\right)$$

$$= -2l + \frac{l}{2}\left(\frac{\eta_1^2}{l^2} + \frac{\eta_2^2}{l^2}\right)$$

$$= -2l + \frac{1}{2}\left(\frac{\eta_1^2}{l} + \frac{\eta_2^2}{l}\right)$$

$$\Rightarrow \dot x_1 = \dot\eta_1$$

$$\dot x_2 = \dot\eta_1 + \dot\eta_2$$

$$\dot y_1 = \frac{1}{2l}\,2\eta_1\dot\eta_1 = \frac{1}{l}\eta_1\dot\eta_1$$

$$\dot y_2 = \frac{1}{2l}\left(2\eta_1\dot\eta_1 + 2\eta_2\dot\eta_2\right) = \frac{1}{l}\left(\eta_1\dot\eta_1 + \eta_2\dot\eta_2\right)$$

$$L = \frac{1}{2}m_1\left[\dot x_1^2 + \dot y_1^2\right] + \frac{1}{2}m\left[\dot x_2^2 + \dot y_2^2\right] - mgy_1 - m_2 g y_2$$

$$= \frac{1}{2}m_1\left[\dot\eta_1^2 + \underbrace{\cancel{\frac{1}{l^2}\eta_1^2\dot\eta_1^2}}_{\text{small}}\right] + \frac{1}{2}m_2\left[(\dot\eta_1 + \dot\eta_2)^2 + \underbrace{\cancel{\frac{1}{l^2}\left(\eta_1\dot\eta_1 + \eta_2\dot\eta_2\right)^2}}_{\text{small}}\right]$$

$$- m_1 g\left(\underbrace{\cancel{-l}}_{\text{constant}} + \frac{1}{2l}\dot\eta_1^2\right) - m_2 g\left(\underbrace{\cancel{-2l}}_{\text{constant}} + \frac{1}{2}\left(\frac{\eta_1^2}{l} + \frac{\eta_2^2}{l}\right)\right)$$

$$\Rightarrow L = \frac{1}{2}m_1\dot\eta_1^2 + \frac{1}{2}m_2(\dot\eta_1 + \dot\eta_2)^2 - \frac{m_1 g}{2l}\dot\eta_1^2 - \frac{m_2 g}{2l}\left(\eta_1^2 + \eta_2^2\right)$$

$$= \frac{1}{2}m_1\dot\eta_1^2 + \frac{1}{2}m_2(\dot\eta_1 + \dot\eta_2)^2 - \frac{g}{2l}\left[(m_1 + m_2)\eta_1^2 + m_2\eta_2^2\right]$$

b) $L = \frac{1}{2} m_1 \dot{\eta}_1^2 + \frac{1}{2} m_2 (\dot{n}_1 + \dot{n}_2)^2 - \frac{g}{2l}\left[(m_1+m_2) n_1^2 + m_2 n_2^2\right]$

We solve Lagrange's equations for n_1 and n_2.

$\underline{n_1:}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{n}_1}\right) = \frac{d}{dt}\left(m\dot{n}_1 + m_2(\dot{n}_1 + \dot{n}_2)\right) = m_1\ddot{n}_1 + m_2(\ddot{n}_1 + \ddot{n}_2)$$

$$= \frac{\partial L}{\partial n_1} = -\frac{g}{\cancel{2}l} \cancel{2}(m_1+m_2)n_1 = \frac{-g}{l}(m_1+m_2)n_1$$

$$\Rightarrow \quad m_1\ddot{n}_1 + m_2(\ddot{n}_1 + \ddot{n}_2) = -\frac{g}{l}(m_1+m_2)n_1 \qquad \underline{\text{EOM 1}}$$

$\underline{n_2:}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{n}_2}\right) = \frac{d}{dt}\left(m_2\dot{n}_1 + m_2\dot{n}_2\right) = m_2(\ddot{n}_1 + \ddot{n}_2)$$

$$\frac{\partial L}{\partial n_2} = \frac{-g}{l} m_2 n_2$$

$$m_2(\ddot{n}_1 + \ddot{n}_2) = \frac{-g}{l} m_2 n_2 \qquad \underline{\text{EOM 2}}$$

Guess $n_1 = c_1 e^{i\omega t} \qquad n_2 = c_2 e^{i\omega t}$

$$\Rightarrow \begin{cases} -m_1\omega^2 c_1 + m_2(-\omega^2 c_1 - c_2\omega^2) = \frac{-g}{l}(m_1+m_2)c_1 \\ -\cancel{m_2}\omega^2(c_1+c_2) = \frac{-g}{l}\cancel{m_2} c_2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \omega^2 - g/l & \frac{m^2\omega}{m_1+m_2} \\ \omega^2 & \omega^2 - \frac{g}{l} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

Let $\gamma = \sqrt{\frac{m_2}{m_1+m_2}}$

$$\Rightarrow \omega^4 - 2\frac{g}{l}\omega^2 + \frac{g^2}{l^2} - \gamma^2\omega^4 = 0$$

$$\omega^4(1-\gamma^2) - 2\frac{g}{l}\omega^2 + \frac{g^2}{l^2} = 0$$

c) Choose EOM since EOM should be equivalent in eigen-problem

$$-\omega^2(c_1+c_2) = -\frac{g}{\ell}c_2$$

$$\Rightarrow -\omega^2 c_1 - \left(\omega^2 + \frac{g}{\ell}\right)c_2 = 0 \quad ; \quad \omega^2 = \frac{g\,[1 \pm \gamma]}{\ell(1-\gamma)^2}$$

$$-\cancel{\frac{g}{\ell}}\frac{[1\pm\gamma]}{(1-\gamma^2)}c_1 - \left(\cancel{\frac{g}{\ell}}\frac{[1\pm\gamma]}{(1-\gamma)^2} + \cancel{\frac{g}{\ell}}\right)c_2 = 0$$

$$\frac{[1\pm\gamma]c_1}{(1-\gamma^2)} = \left(\frac{[1\pm\gamma]}{(1-\gamma)^2} - 1\right)c_2$$

$$c_1 = \left(1 - \frac{(1-\gamma^2)}{1\pm\gamma}\right)c_2 = (1 - 1 \pm \gamma)c_2$$

$$c_1 = \pm\gamma c_2$$

Recall that $\gamma = \sqrt{\frac{m_2}{m_1+m_2}}$. If $\frac{m_1}{m_2}$ large, $m_1 \gg m_2$

$\gamma \to 0$, $\omega \to \sqrt{\frac{g}{\ell}}$. Only bottom pendulum moves.

If $m_2 \gg m_1 \Rightarrow \gamma \to 1 \Rightarrow \omega = \frac{\sqrt{\frac{g}{\ell}}}{0}$. No oscillation.

or $\omega = \sqrt{\frac{g}{\ell}}\frac{1}{\sqrt{2}} = \sqrt{\frac{g}{2\ell}} \Rightarrow$ pendulum of length 2ℓ

$$\Rightarrow \omega^2 = \frac{2\frac{g}{l} \pm \sqrt{4\frac{g^2}{l^2} - 4(1-\gamma^2)\frac{g^2}{l^2}}}{2(1-\gamma)^2}$$

$$\Rightarrow \omega^2 = \frac{\cancel{2}g\cancel{l}}{\cancel{2}(1-\gamma)^2 l} \pm \frac{\cancel{2}\frac{g}{l}\sqrt{1-1+\gamma^2}}{\cancel{2}(1-\gamma)^2}$$

$$= \frac{g}{(1-\gamma)^2 l}\left[1 \pm \sqrt{\gamma^2}\right]$$

$$= \frac{g}{l(1-\gamma^2)}\left[1 \pm \gamma\right] = \frac{g}{l}\ (1 \pm \gamma)^{-1}$$

4.4)

$$\mathcal{L} = \frac{1}{2} m\left(a^2\dot\theta^2 + a^2\sin^2\theta\,\Omega^2\right) + mga\cos\theta$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial\dot\theta} - \frac{\partial \mathcal{L}}{\partial\theta} = ma^2\ddot\theta - ma^2\sin\theta\cos\theta\,\Omega^2 + mga\sin\theta = 0$$

since $\frac{dH}{dt} = \int dt\,\dot q\left[\frac{dp}{dt} - \frac{\partial \mathcal{L}}{\partial q}\right] = 0$

take $\int_0^t dt\,\dot\theta\left[ma^2\ddot\theta - ma^2\sin\theta\cos\theta\,\Omega^2 + mga\sin\theta\right] = 0$

$$= \int_0^t dt\,\frac{d}{dt}\left(\frac{ma^2}{2}\dot\theta^2 - \frac{ma^2}{2}\Omega^2\sin^2\theta - \frac{ma^2\Omega^2}{2}\sin^2\theta - mga\cos\theta\right)$$

$$\Rightarrow \frac{ma^2}{2}\dot\theta^2 - \frac{ma^2\Omega^2\sin^2\theta}{2} - mga\cos\theta = \text{const.} \quad (1)$$

b) In equilibrium $\dot\theta = \ddot\theta = 0$

(1) $\Rightarrow ma^2\sin\theta\cos\theta\,\Omega^2 = mga\sin\theta \quad (2)$

$\Rightarrow \cos\theta = g/a\Omega^2$ if $\sin\theta \neq 0$

or $\sin\theta = 0$

$\Rightarrow \theta_0 = 0, \pi,$ or $\cos\theta_0 = g/a\Omega^2$

Case I $\quad \theta_0 = 0 \Rightarrow \ddot\theta = \sin\theta\cos\theta\,\Omega^2 - \frac{g\sin\theta}{a}$

$$= \theta(\Omega^2 - g/a)$$

$\Rightarrow -\omega^2 = -\Omega^2 + g/a$, stable for $-\omega^2 > 0$

$\Rightarrow |\Omega| \le \sqrt{g/a}$

case II $\theta_0 = \pi$

 $\Rightarrow \ddot{\theta} = \theta(-\Omega^2 - g/a)$

$\Rightarrow -\omega^2 = -(\Omega^2 + g/a) \geq 0$

which is never true.

case III, $\cos\theta_0 = g/a\Omega^2$, $\cos^2\theta_0 = \left(\frac{g}{a\Omega^2}\right)^2$

$\sin^2\theta_0 = 1 - (g/a\Omega^2)^2$

consider $\theta = \theta_0 + \delta$ with δ a small variation

note $\sin(u+v) = \sin u \cos v + \cos u \sin v$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$\Rightarrow \sin(\theta_0 + \delta) \cong \sin\theta_0 + \delta\cos\theta_0$

$\cos(\theta_0 + \delta) \cong \cos\theta_0 - \delta\sin\theta_0$

$$\Rightarrow \ddot{\theta} = \ddot{\delta} = (\sin\theta_0 + \delta\cos\theta_0)(\cos\theta_0 - \delta\sin\theta_0)\Omega^2 - \frac{g}{a}(\sin\theta_0 + \delta\cos\theta_0)$$

$$= (\sin\theta_0\cos\theta_0 + \delta(\cos^2\theta_0 - \sin^2\theta_0))\Omega^2 - \frac{g}{a}(\sin\theta_0 + \delta\cos\theta_0)$$

$$= \Omega^2\left(\sqrt{1-(g/a\Omega^2)^2}\,\frac{g}{a\Omega^2} - \delta\right) - \frac{g}{a\Omega^2}\Omega^2\left(\sqrt{1-\left(\frac{g}{a\Omega^2}\right)^2} + \delta\left(\frac{g}{a\Omega^2}\right)\right)$$

$$= \delta\left(-\Omega^2 + g^2/a^2\Omega^2\right)$$

$\Rightarrow -\omega^2 = \Omega^2 - \frac{g^2}{a^2\Omega^2} \geq 0;\ \Omega \geq \sqrt{g/a}$

compare the equation for stable osc (2)

to $F_T = 0 = F_{Inertial} + F_{coriolis} + F_{centrifugal}$

$$= F_I - 2m\vec{\Omega}\times\dot{\vec{r}} - m\vec{\Omega}\times\vec{\Omega}\times\vec{r}$$

$$= (mg\sin\theta - m\Omega^2 a\cos\theta\sin\theta)\hat{\theta} \text{ so } 0$$

$$\Rightarrow \sin\theta\, mga = ma^2\Omega^2\sin\theta\cos\theta$$

c) $p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta} \Rightarrow \dot{\theta} = p_\theta/ma^2$

so $H = p_\theta\dot{\theta} - L$

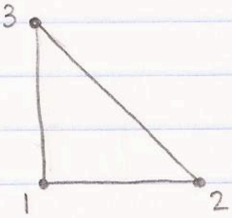
$$= \frac{p_\theta^2}{2m} - \frac{1}{2}ma^2\sin^2\theta\,\Omega^2 - mga\cos\theta$$

which is the result of part a)

so this is constant. It is not T+V.

CHAITANYA
MURTHY

4.9.(a)



$m_1 = m_2 = m_3 = m$

$k_{12} = k_{13} = k_{23} = k$

$V = \frac{1}{2}k\left(\delta l_{12}^2 + \delta l_{13}^2 + \delta l_{23}^2\right)$

All of the relative distances, and hence V, are invariant under uniform translation or rotation of the molecule. These three degrees of freedom (two translational and one rotational) are the $\omega^2 = 0$ modes.
For convenience, they can be removed by equating to zero the total momentum and the total angular momentum of the molecule.

Let $\vec{x}_a = \vec{r}_a - \vec{r}_a^{\,0}$ be the deviation of atom (a) from its equilibrium position $\vec{r}_a^{\,0}$. Since the center of mass is at rest, we have

$$\sum m_a \vec{r}_a = \text{constant} = \sum m_a \vec{r}_a^{\,0} \Rightarrow \sum m_a \vec{x}_a = 0$$

This gives two relations:

① $m(x_1 + x_2 + x_3) = 0 \Rightarrow x_1 = -(x_2 + x_3)$

② $m(y_1 + y_2 + y_3) = 0 \Rightarrow y_1 = -(y_2 + y_3)$

For small oscillations, the angular momentum can be written as

$$\vec{M} = \sum m_a \vec{r}_a \times \vec{v}_a \approx \sum m_a \vec{r}_a^{\,0} \times \dot{\vec{x}}_a = \frac{d}{dt}\left(\sum m_a \vec{r}_a^{\,0} \times \vec{x}_a\right)$$

The condition for this to be zero during the small oscillations is

$$\sum m_a \vec{r}_a^{\,0} \times \vec{x}_a = 0$$

Choosing the origin at $\vec{r}_1^{\,0}$ (it can be chosen arbitrarily), we obtain

③ $m(y_2 - x_3) = 0 \Rightarrow y_2 = x_3$

To lowest order, the changes in the interparticle distances are given by

$$\delta l_{12} = x_2 - x_1$$

$$\delta l_{13} = y_3 - y_1$$

$$\delta l_{23} = \frac{1}{\sqrt{2}}(x_2 - x_3) + \frac{1}{\sqrt{2}}(y_3 - y_2)$$

Here δl_{AB} is simply the component along the line joining A and B of the vector $\vec{x}_B - \vec{x}_A \ldots$

The Lagrangian is

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 \right) - \frac{1}{2} k \left(\delta l_{12}^2 + \delta l_{13}^2 + \delta l_{23}^2 \right)$$

Using relations ①, ② and ③ to eliminate x_1, y_1 and y_2:

$$L = \tfrac{1}{2} m \left[(\dot{x}_2 + \dot{x}_3)^2 + \dot{x}_2^2 + 2\dot{x}_3^2 + (\dot{x}_3 + \dot{y}_3)^2 + \dot{y}_3^2 \right]$$
$$- \tfrac{1}{2} k \left[(2x_2 + x_3)^2 + (2y_3 + x_3)^2 + \tfrac{1}{2}(x_2 - 2x_3 + y_3)^2 \right]$$

The equations of motion for the three remaining variables are:

$$(x_2): \quad m(2\ddot{x}_2 + \ddot{x}_3) = -\frac{k}{2}\left[4(2x_2 + x_3) + (x_2 - 2x_3 + y_3) \right]$$
$$(x_3): \quad m(\ddot{x}_2 + 4\ddot{x}_3 + \ddot{y}_3) = -\frac{k}{2}\left[2(2x_2 + x_3) + 2(2y_3 + x_3) - 2(x_2 - 2x_3 + y_3) \right]$$
$$(y_3): \quad m(\ddot{x}_3 + 2\ddot{y}_3) = -\frac{k}{2}\left[4(2y_3 + x_3) + (x_2 - 2x_3 + y_3) \right]$$

Looking for solutions of the form $\vec{x}_a = \vec{x}_a^{\,0} e^{i\omega t}$,

$$\frac{2m\omega^2}{k} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2^0 \\ x_3^0 \\ y_3^0 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 1 \\ 2 & 8 & 2 \\ 1 & 2 & 9 \end{bmatrix} \begin{bmatrix} x_2^0 \\ x_3^0 \\ y_3^0 \end{bmatrix}$$

This is the eigenvalue equation. It has a non-trivial solution when

$$\begin{vmatrix} 9-4\lambda & 2-2\lambda & 1 \\ 2-2\lambda & 8-8\lambda & 2-2\lambda \\ 1 & 2-2\lambda & 9-4\lambda \end{vmatrix} = 0, \quad \text{where } \lambda = \frac{m\omega^2}{k}$$

$$\Rightarrow (9-4\lambda)^2(8-8\lambda) + 2(2-2\lambda)^2 - (8-8\lambda) - 2(2-2\lambda)^2(9-4\lambda) = 0$$
$$\Rightarrow 8(1-\lambda)\left[(9-4\lambda)^2 + (1-\lambda) - 1 - (1-\lambda)(9-4\lambda) \right] = 0$$
$$\Rightarrow (1-\lambda)\left[16\lambda^2 - 72\lambda + 81 - \lambda - (4\lambda^2 - 13\lambda + 9) \right] = 0$$
$$\Rightarrow (1-\lambda)(12\lambda^2 - 60\lambda + 72) = 0 \quad \Rightarrow \quad 12(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

So the eigenfrequencies are:

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{2k}{m}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{3k}{m}}$$

Problem 5: (4.10)

$$\underline{V} \equiv \begin{pmatrix} V & V_{12} \\ V_{12} & V \end{pmatrix} \qquad \underline{m} \equiv \begin{pmatrix} m & m_{12} \\ m_{12} & m \end{pmatrix}$$

(a) Solve for eigenvalues

$$(\underline{V} - \omega^2 \underline{m})\rho = 0$$

$$\det \begin{pmatrix} V-\omega^2 m & V_{12}-\omega^2 m_{12} \\ V_{12}-\omega^2 m_{12} & V-\omega^2 m \end{pmatrix} = 0$$

$$(V-\omega^2 m)^2 - (V_{12}-\omega^2 m_{12})^2 = 0$$

$$V-\omega^2 m = \pm V_{12} \mp \omega^2 m_{12}$$

$$V \mp V_{12} = \omega^2 (m \mp m_{12})$$

$$\omega^2 = \frac{V \mp V_{12}}{m \mp m_{12}}$$

$$\boxed{\omega_1^2 = \frac{V-V_{12}}{m-m_{12}} \qquad \omega_2^2 = \frac{V+V_{12}}{m+m_{12}}}$$

From this:

$(V, m) \to 0$ $\omega_1^2 = \omega_2^2 = \frac{V_{12}}{m_{12}}$

$(V_{12}, m_{12}) \to 0$, $\omega_1^2 = \omega_2^2 = \frac{V}{m}$

Solve for Eigenvectors

For ω_1^2 we get

$$\left[V - \frac{(V-V_{12})}{(m-m_{12})} m\right]\rho_1^{(1)} + \left[V_{12} - \frac{(V-V_{12})}{(m-m_{12})} m_{12}\right]\rho_2^{(1)} = 0$$

$$\frac{(\cancel{Vm} - Vm_{12} - \cancel{Vm} + V_{12}m)}{(m-m_{12})}\rho_1^{(1)} + \frac{(V_{12}m - \cancel{V_{12}m_{12}} - Vm_{12} + \cancel{V_{12}m_{12}})}{(m-m_{12})}\rho_2^{(1)} = 0$$

$$\rho_1^{(1)} = -\rho_2^{(1)} \Rightarrow \boxed{\rho^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For ω_2^2 we get

$$\left[V - \frac{(V+V_{12})}{(m+m_{12})} m\right] \rho_1^{(2)} + \left[V_{12} - \frac{(V+V_{12})}{(m+m_{12})} m_{12}\right] \rho_2^{(2)} = 0$$

$$\frac{(\cancel{Vm} + Vm_{12} - \cancel{Vm} - V_{12}m)}{(m+m_{12})} \rho_1^{(2)} + \frac{(V_{12}m + \cancel{V_{12}m_{12}} - Vm_{12} - \cancel{V_{12}m_{12}})}{(m+m_{12})} \rho_2^{(2)} = 0$$

$$\therefore \rho_1^{(2)} = \rho_2^{(2)} \rightarrow \rho^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normalizing them according to $\sum_\lambda \sum_\sigma \rho_\sigma^{(t)} m_{\sigma\lambda} \rho_\lambda^{(s)} = \delta_{st}$

gives

$$\boxed{\rho^{(1)} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \rho^{(2)} = \frac{1}{\sqrt{2(m+m_{12})}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

(b) As $(m_{12}, V_{12}) \to 0$

$$\begin{pmatrix} V-\omega^2 m & 0 \\ 0 & V-\omega^2 m \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = 0 \qquad \omega^2 = \frac{V}{m}$$

Eigenvectors

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = 0$$

we are free to choose any two Eigenvectors which can be linearly independent. $\rho^{(1)}$ & $\rho^{(2)}$ with solutions of the form $\underline{z}^{(s)} = e^{i\phi_s} \rho^{(s)}$

(c) Say we have two linearly independent eigenvectors $\tilde{\rho}^{(1)}$ & $\tilde{\rho}^{(2)}$. According the Gram-Schmidt orthogonalization procedure, for two vectors v_1, v_2, the new basis vectors are...

$$\vec{u}_1 = \frac{\vec{v}_1}{\langle \vec{v}_1 | \vec{v}_1 \rangle} \quad \& \quad \vec{q} = \vec{v}_2 - \frac{\langle \vec{v}_2 | \vec{v}_1 \rangle}{\langle \vec{v}_1 | \vec{v}_1 \rangle} \vec{v}_1$$

$$\vec{u}_2 = \frac{\vec{q}}{\langle \vec{q} | \vec{q} \rangle}$$

Here, we defined the inner product and normalization to $\langle \tilde{\rho} | \underline{m} | \tilde{\rho} \rangle$. So following Gram Schmidt.

Definition:

$$\langle \tilde{\rho}^{(s)} | \underline{m} | \tilde{\rho}^{(t)} \rangle = \sum_{\lambda} \sum_{\sigma} \tilde{\rho}^{(s)}_{\sigma} m_{\sigma\lambda} \tilde{\rho}^{(t)}_{\lambda}$$

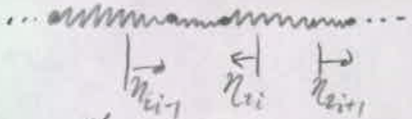
$$\rho^{(1)} = \frac{\tilde{\rho}^{(1)}}{\langle \tilde{\rho}^{(1)} | \underline{m} | \tilde{\rho}^{(1)} \rangle}$$

$$\rho^{(2)} = C_2 \underbrace{\left(\tilde{\rho}^{(2)} - \frac{\langle \tilde{\rho}^{(2)} | \underline{m} | \tilde{\rho}^{(1)} \rangle}{\langle \tilde{\rho}^{(1)} | \underline{m} | \tilde{\rho}^{(1)} \rangle} \tilde{\rho}^{(1)} \right)}_{q^{(2)}}$$

$$\boxed{\begin{aligned} C_2 &= \frac{1}{\langle q^{(2)} | \underline{m} | q^{(2)} \rangle} \leftarrow q^{(2)} \\ C_1 &= \frac{1}{\langle \tilde{\rho}^{(1)} | \underline{m} | \tilde{\rho}^{(1)} \rangle} \end{aligned}}$$

(2)

4.12



$$U_i = \sum_{i=1}^{N} k_1(\eta_{2i}-\eta_{2i-1})^2 + k_2(\eta_{2i}-\eta_{2i+1})^2$$

$$\Rightarrow L = \sum_i^{2N} \tfrac{1}{2} m\dot{\eta}_i^2 - \sum \tfrac{1}{2}k_1(\eta_{2i}-\eta_{2i-1})^2 + \tfrac{1}{2}k_2(\eta_{2i}-\eta_{2i+1})^2$$

evens: $$\frac{\partial L}{\partial \eta_{2i}} = -\Big[\big(k_1(\eta_{2i}-\eta_{2i-1}) + k_2(\eta_{2i}-\eta_{2i+1})\big)\Big]$$

odds: $$\frac{\partial L}{\partial \eta_{2j+1}} = -\frac{1}{2}\left\{\frac{\partial}{\partial \eta_{2j+1}}\Big[k_1(\eta_{2i+2}-\eta_{2i+1})^2 + k_2(\eta_{2i}-\eta_{2i+1})^2\Big]\right\}$$

$$= -\Big[-k_1(\eta_{2i+2}-\eta_{2i+1}) - k_2(\eta_{2i}-\eta_{2i+1})\Big]$$

$\Rightarrow$ EOMs:

$$\boxed{\begin{aligned} &\text{evens: } m\ddot{\eta}_{2j} = k_1(\eta_{2i}-\eta_{2i-1}) + k_2(\eta_{2i}-\eta_{2i+1}) \\ &\text{odds: } m\ddot{\eta}_{2j+1} = k_1(\eta_{2i+1}-\eta_{2i+2}) + k_2(\eta_{2i+1}-\eta_{2i}) \end{aligned}}$$

Now take $\eta_{2j} = \alpha e^{i(2jqa)-\omega t}$ & $\eta_{2j+1} = \beta e^{i(2j+1)qa-\omega t}$

in evens: $-\omega^2 m\alpha = k_1(\alpha e^{(\dots)} - \beta e^{-iqa}e^{(\dots)}) + k_2(\alpha e^{(\dots)} - \beta e^{(\dots)} e^{iqa})$

$$-\omega^2 m\alpha = k_1(\alpha - \beta e^{-iqa}) + k_2(\alpha - \beta e^{iqa})$$

$$\boxed{\alpha(\omega^2 m + k_1 + k_2) = \beta(k_1 e^{-iqa} + k_2 e^{iqa})}$$

odds: $-\omega^2 m\beta = k_1(\beta - \alpha e^{+iqa}) + k_2(\beta - \alpha e^{+iqa})$

$$\begin{cases} (\omega^2 m + k_1 + k_2) = \frac{\alpha}{\beta}\left(k_1 e^{iqa} + k_2 e^{-iqa}\right) \\ \omega^2 m + k_1 + k_2 = \frac{\beta}{\alpha}\left(k_1 e^{-iqa} + k_2 e^{iqa}\right) \end{cases}$$ — invert, then divide eqns.

$$(\omega^2 m + k_1 + k_2)^2 = (k_1 e^{iqa} + k_2 e^{-iqa})(k_1 e^{-iqa} + k_2 e^{iqa})$$
$$= k_1^2 + k_1 k_2 e^{2iqa} + k_1 k_2 e^{-2iqa} + k_2^2$$
$$= k_1^2 + k_2^2 + 2k_1 k_2 (\cos 2qa)$$

$$\Rightarrow \boxed{\omega^2 m = \sqrt{k_1^2 + k_2^2 + 2k_1 k_2 (\cos 2qa)} - k_1 - k_2}$$

Dispersion relation.

if $k_1 = k_2$, $\omega^2 m = \sqrt{2k^2 + 2k^2\cos(2qa)} - 2k$

(b) Using Periodic boundary conditions:

$$\eta_{2j}(aj) = \eta_{2j}(aj + 2Na)$$
$$e^{2iNaq} = 1$$
$$\Rightarrow \boxed{q = \frac{n\pi}{2Na}}$$

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Phys 200A
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Problem Set 5

Problem 7 (Fetter and Walecka 4.13)

Refer to Figure 23.1 on page 101 of Fetter and Walecka for a picture. We have a long chain of identical pendulums connected by springs. In equilibrium, all the pendulums are vertical and the springs unstretched at length a. Therefore a is the horizontal distance separating the tops of the pendulums. Define θ_i as the angle from the vertical for the ith pendulum, and η_i as the transverse displacement from the equilibrium position. Now we can write a Lagrangian. Using the small oscillation approximation, we will express everything in terms of η_i.

For small displacements, we can see from the picture that

$$\frac{\eta_i}{l} = \sin\theta_i \approx \theta_i.$$

This also gives

$$\dot{\eta_i} = l\dot{\theta}_i.$$

Using these approximations, we obtain:

$$T = \sum_{i=1}^{N} \frac{1}{2} m v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m (l\dot{\theta}_i)^2 = \sum_{i=1}^{N} \frac{1}{2} m \dot{\eta_i}^2.$$

$$V_{grav} = \sum_{i=1}^{N} mgl(1-\cos\theta_i) = \sum_{i=1}^{N} mgl\left(\frac{1}{2}\theta_i^2\right) = \sum_{i=1}^{N} mgl\left(\frac{\eta_i}{2l^2}\right) = \sum_{i=1}^{N} \frac{mg}{2l}\eta_i^2.$$

$$V_{spr} = \sum_{i=1}^{N} \frac{1}{2} k(\eta_{i+1} - \eta_i)^2.$$

Note: The displacement from equilibrium of a spring is $(d-d_0)_i = (\eta_{i+1} - \eta_i) + O(\eta^4)$, so we simply ignore the higher order terms since η_i is small.

Now we have our Lagrangian completely in terms of the coordinates η_i.

$$L = \sum_{i=1}^{N} \frac{1}{2} m\dot{\eta_i}^2 - \frac{1}{2} k(\eta_{i+1} - \eta_i)^2 - \frac{mg}{2l}\eta_i^2.$$

We obtain the equations of motion for each η_i:

$$m\ddot{\eta_i} + \left(2k + \frac{mg}{l}\right)\eta_i - k(\eta_{i+1} + \eta_{i-1}) = 0.$$

As suggested in Fetter and Walecka (p.115), the reasonable solution to try is

$$\eta_j = Ae^{i(qx_j - \omega t)},$$

where $x_j = ja$ and A is some constant. This is the plane wave normal mode with wavenumber q and frequency ω that propagates along the chain. Plugging this in to the equation of motion yields

$$-m\omega^2\eta_j + \left(2k + \frac{mg}{l}\right)\eta_j - k(e^{iqa} + e^{-iqa})\eta_j = 0.$$

Since $\eta_j \neq 0$, we have

$$\begin{aligned}\omega^2 &= \frac{g}{l} + \frac{2k}{m}\left(1 - \cos(qa)\right) \\ &= \frac{g}{l} + \frac{2k}{m}\left(2\sin^2\left(\frac{qa}{2}\right)\right) \\ &= \frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right).\end{aligned}$$

This gives us the dispersion relation

$$\omega_+(q) = \sqrt{\frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)}$$

$$\omega_-(q) = -\sqrt{\frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)}.$$

See attached sheet for plots of the dispersion relation.

If we assume periodic boundary conditions, we put restrictions on q. Periodic boundary conditions require that $\eta_j = \eta_{j+N}$ for all j. Using the plane wave definition for η_j, we have the requirement

$$e^{iqNa} = 1 \Rightarrow q = \frac{2\pi n}{Na}$$

for some $n \in \mathbb{Z}$.

Now the allowed frequencies are

$$\omega^2(n) = \frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{\pi n}{N}\right).$$

We can see that if $n = 0$ we have the lowest frequency

$$\omega_0^2 = \frac{g}{l},$$

which is the oscillation frequency of a single pendulum without any springs attached. This frequency corresponds to the physical situation of all the pendulums swinging in unison at frequency $\sqrt{g/l}$.