⇒ Solving the Wave Equation for String

a.) Exact/General

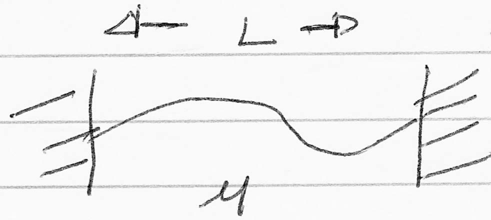
Note 2 approaches:

i.) Fourier Series → "Bernoulli's Solution"
ii.) Pulse Tracking → "D'Alembert's Solution"

i.) Fourier Series

eigenfctns are $\sin \frac{n\pi x}{L}$

if standard case:



$$\frac{1}{c^2}\partial_{tt} u = \partial_{xx} u$$

General Fourier Series Soln

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \rho_n(x)\, C_n \cos(\underbrace{\omega_n}_{\frac{n\pi c}{L}} t + \overbrace{\phi_n}^{\text{phase}})$$

$$\rho_n = \left(\frac{2}{L\mu}\right)^{1/2} \sin \frac{n\pi x}{L}$$

where $\int_0^L \rho_n(x)\,\rho_m(x)\,\underbrace{\mu}_{\text{mass matrix (recall osc.)}} = \delta_{m,n}$

So, to determine motion for all times:

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \dot{u}(x,0) &= g(x) \end{aligned}\right\} \Rightarrow C_n,\ \phi_n$$

i.e. re-write $u(x,t)$ as :

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L\mu}\right)^{1/2} \sin\frac{n\pi x}{L}\left[a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right]$$

i.e. $a_n = C_n \cos\phi_n$

$b_n = -C_n \sin\phi_n$

$$\Rightarrow \quad a_n = \left(\frac{2}{L\mu}\right)^{1/2} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, \mu \, dx$$

$$\frac{n\pi c}{L}\, b_n = \left(\frac{2}{L\mu}\right)^{1/2} \int_0^L \sin\frac{n\pi x}{L}\, g(x)\, \mu \, dx$$

and, $\left.\begin{matrix} C_n, \phi_n \\ a_n, b_n \end{matrix}\right\} \Rightarrow$ normal coordinates

i.e. generalized amplitudes

$$\underbrace{q_n}_{\text{normal coordinate}} = C_n\left(\omega_n t + \phi_n\right)$$

$\Rightarrow$

$$L = \tfrac{1}{2} \sum_n \left(\dot{c}_n^2 - \omega_n^2 c_n^2 \right)$$

$$H = \tfrac{1}{2} \sum_n \left(\dot{c}_n^2 + \omega_n^2 c_n^2 \right)$$

$$= \tfrac{1}{2} \sum_n \omega_n^2 C_n^2 .$$

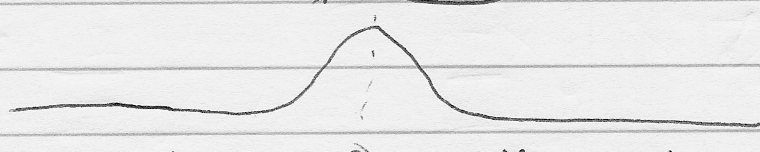
$\rightsquigarrow$ the usual.....

b.) Pulse Dynamics ~~f~~

→ consider infinite string first, for simplicity

pluck

→ if i.c.



?

$x+ct$ $\qquad$ $x-ct$

⇒

no need for Fourier integral!

∴ natural to change variables to:

$$x \qquad r \equiv x - ct \qquad \to \text{propagating} \to +\infty$$

$$\to$$

$$t \qquad s \equiv x + ct \qquad \to \text{propagating} \to -\infty ,$$

$$u(x,t) \equiv U(r,s)$$

Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial r \partial s} + \frac{\partial^2 u}{\partial s^2}$$

similarly

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\overbrace{\frac{\partial r}{\partial t}}^{-c} + \frac{\partial u}{\partial s}\overbrace{\frac{\partial s}{\partial t}}^{+c}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} - 2\frac{\partial^2 u}{\partial r \partial s} + \frac{\partial^2 u}{\partial s^2}$$

$\Rightarrow$ wave eqn. becomes:

$$4\frac{\partial^2 u}{\partial r \partial s} = 0$$

$\rightarrow$ simple representation (suggests pulse variables "natural")

Thus:

$\frac{\partial}{\partial r}\frac{\partial u}{\partial s} = 0$, so integrating w/r r gives

$$\frac{\partial u}{\partial s} = \phi'(s)$$

arbitrary fctn indep. of r

$\Rightarrow$

$$\boxed{u = \phi(s) + \psi(r)}$$ general solution

∴ $u(x, t) = \phi(x+ct) + \psi(x-ct)$

soln. as super-position of left / rt. moving pulse!

To obtain ϕ, ψ: fit i.c.'s

$$u(x,0) = f(x)$$
$$\dot{u}(x,0) = g(x)$$

consider 2 i.c.'s

$$u(x,0) = f(x)$$
$$\dot{u} = 0$$

and

$$u(x,0) = 0$$
$$\dot{u}(x,0) = g(x)$$

and super-pose.

a) $u_1(x,0) = f(x)$

$\therefore \; u_1(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]$

b) $\dot{u}_2(x,0) = g(x)$

$$u_2(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

so, can superpose u_1, u_2 to obtain D'Alembert's solution:

$$\boxed{u(x,t) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi}$$

But what of finite string?

then boundary conditions enter.

$0 \le x \le L$

$u(0,t) = 0$
$u(L,t) = 0$

as well $u(x,0) = f(x)$
as i.c.'s $\dot{u}(x,0) = g(x)$

Crux of issue:

- to use D'Alembert's solution, need $f(x)$, $g(x)$ for all x (i.e. outside $[0, L]$ because $x \pm ct \to \pm\infty$ as $t \to \infty$!

but

- f, g given on $[0, L]$ only, for finite string!

$\therefore$ need
- extend f, g outside $[0, L]$
- respect B.C.'s.

How?

- if f, g odd about origin $x=0$, $x=L$

D's solution satisfies B.C.'s for all time

i.e. $f(-x) = -f(x)$, $g(-x) = -g(x)$

assures $u(x,t)\Big|_{x=0} = 0$, all time.

similarly : if f, g odd about $x = L$,
$u(L,t) = 0$, all time.

$$f[L + (x-L)] = -f[L - (x-L)]$$

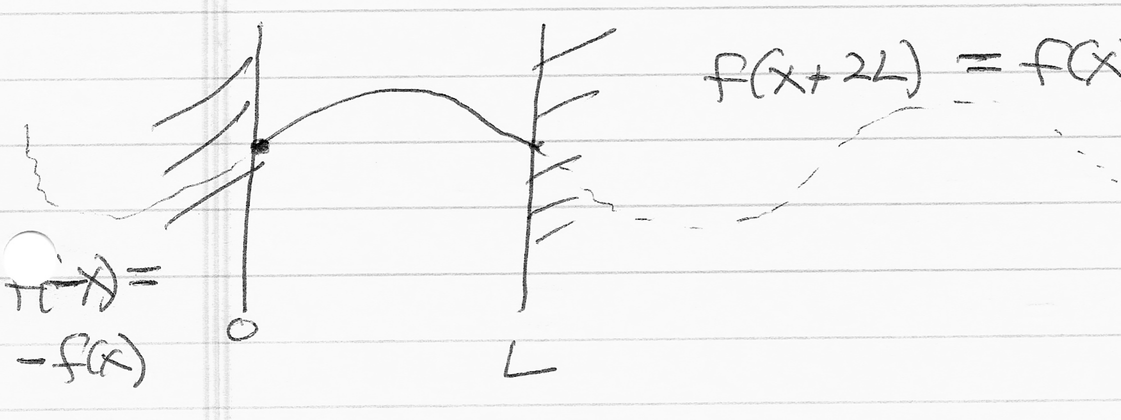
$$g[L + (x-L)] = -g[L - (x-L)]$$

$\therefore$ $$f(x) = -f[2L - x] = f(x-2L) = f(x+2L)$$
$$+\ g(x) = g[2L - x] = g(x-2L) = g(x+2L)$$

so have conditions f, g $\to$ $\begin{cases} \text{odd about } x \\ \text{periodic, with} \\ 2L \text{ period.} \end{cases}$

i.e. $$\left.\begin{aligned} f(-x) &= -f(x) \\ g(-x) &= -g(x) \\ f(x+2L) &= f(x) \\ g(x+2L) &= g(x) \end{aligned}\right\} \to$$ way of extending to obtain "virtual I.C.'s" for D's solution,

$f(x+2L) = f(x)$



$f(-x) = -f(x)$

Note: B's / D's solutions ⇒ equivalent

c.f. F+W: 215 - 219
{Proof rests on properties of Fourier series convergence.

b.) String is classic case of:

$\boxed{\text{Sturm - Liouville Problem}}$,

a well known, well-studied example.

- eigenfunctions expansion (exact)
- Green's Function ⟷ impulse response

and approximation methods:

- variational principle (Rayleigh-Ritz)
- perturbation theory.

Now, to motivate general nature of Sturm-Liouville, add general restoring force $-V(x)\,u(x,t)$

$$\Rightarrow \quad \sigma(x)\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x)\,\frac{\partial u}{\partial x}\right] - V(x)\,u$$

$(\sigma = \mu)$

writing : $u(x,t) = \rho(x)\, e^{-i\omega t}$

$\Rightarrow$ $\boxed{\text{Sturm-Liouville Eqn:} \quad -\frac{d}{dx}\left[\tau(x)\frac{d\rho}{dx}\right] + V(x)\,\rho = \omega^2 \sigma(x)\,\rho \quad \text{on } a < x < b}$

$\tau > 0$
$\sigma > 0$
τ, σ, V real.

Types of b.c.'s.

i.) fixed ends $\qquad \rho(a) = \rho(b) = 0$

ii.) "natural" or free end $\qquad$ (c.f. Lagrange eqns.)

$$\tau \frac{d\rho}{dx}\Big|_a = \tau\frac{d\rho}{dx}\Big|_b = 0$$

iii.) general homogeneous :

$$\alpha \frac{d\rho}{dx} = \beta\rho \Big|_{a,b} \qquad \underline{\text{i.e.}} \quad \underbrace{\frac{1}{\rho}\frac{d\rho}{dx}}_{\text{fixed end slope}} = \frac{\beta}{\alpha}$$

iv.) periodic: $\phi(b) = \phi(\epsilon)$
$\phi'(b) = \phi'(\epsilon)$.

B.C.'s $\Rightarrow$ solutions exist only for certain set of eigenvalues, ω_n^2, $n = 1, 2, \ldots \infty$, with corresponding eigenfunctions ϕ_n.

i.e.

$$-\frac{d}{dx}\left[T(x)\frac{d\phi_n}{dx}\right] + V(x)\,\phi_n(x) = \omega_n^2\,\sigma(x)\,\phi_n(x)$$

Properties of eigenvalues:
eigenfunctions

$\rightarrow$ $\omega_n^2 \rightarrow \infty$, as $n \rightarrow \infty$

$\rightarrow$ $\omega_n^2 \geq 0$ (stability)
($< 0 \rightarrow$ instability).

$\rightarrow$ orthogonality $\rightarrow$ standard proof:

i.e.

$$\frac{d}{dx} T\phi_p' - V\phi_p = -\omega_p^2\,\sigma\,\phi_p \qquad (1)$$

$$\frac{d}{dx} T\phi_q'^{*} - V\phi_q^{*} = -\omega_q^{2}\,\sigma^{*}\phi_q \qquad (2)$$

(1) $\otimes\, \rho_q^* \;-\; 2\, \otimes\, \rho_p \Rightarrow$

$$\rho_q^* \frac{d}{dx} \tau \rho_p' - \rho_p \frac{d}{dx} \tau \rho_q'^* - \cancel{\rho_q^* y \rho_p} + \cancel{\rho_p y \rho_q^*}$$

$$= (\omega_q^{*2} - \omega_p^2)\, \rho_q^* \tau \rho_p$$

integrating $\Rightarrow$

$$\int_a^b \frac{d}{dx}\left(\rho_q^* \tau \rho_p' - \rho_p \tau \rho_q'^*\right) =$$

$$(\omega_q^{2*} - \omega_p^2) \int_a^b \rho_q^* \tau \rho_p$$

but LHS $\to 0$, <u>all</u> b.c.'s. $\left\{ \text{terms cancel on pull through} \right.$

$$\Rightarrow ((\omega_q^2)^* - \omega_p^2) \int_a^b \rho_q^* \,\tau \rho_p\, dx = 0$$

$$\therefore \int_a^b \rho_p^*(x)\, \rho_q(x)\, \tau\, dx = \delta_{p,q}$$

$\to p \neq q \quad (\omega_q^2)^* \neq \omega_p^2 \;\to\; \int \; = 0$

$\Rightarrow \quad p = q$

$$\omega_q^{2*} - \omega_p^2 \Rightarrow -2\,\mathrm{Im}\,\omega_p^2$$

$$-2\,\mathrm{Im}\,\omega_p^2 \int_a^b d\tau\, \mu\, \rho_p^* \rho_p = 0$$

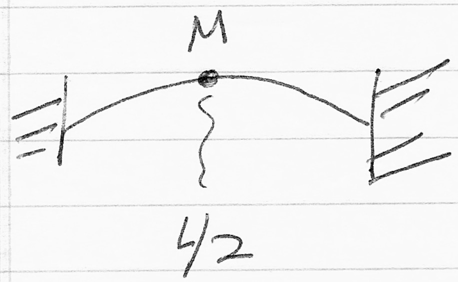
$$\mathrm{Im}\,\omega_p^2 = 0$$

$\Rightarrow$ eigenvalues real.

- Completeness – proved with variational prin.

e.g.

see next 2 pg.



(problem 4.17)

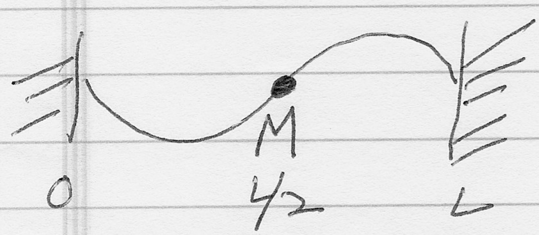
odd modes (spatial) $\Rightarrow$ m not moving

$$\Rightarrow \frac{\omega_n}{c} = \frac{n\pi}{L} \qquad n = 2, 4, \ldots$$

even modes $\Rightarrow$ mass moves.

Example pblm.

→ Modes of string fixed at 0, L with $c^2 = T/\mu$ and mass M at center.



Now $\left[\mu + M\,\delta(x - L/2)\right] y_{tt} = \partial_x T y_x$

$$\Rightarrow \int_{L/2_-}^{L/2_+} \left(\mu + M\,\delta(x-L/2)\right) y_{tt} = T y_x \Big|_{L/2_-}^{L/2_+}$$

$$-M\omega^2 y(L/2) = T\left(y_x\Big|_{L/2_+} - y_x\Big|_{L/2_-} \right)$$

and continuity.

Now, must satisfy the boundary conditions:

$$y(0) = y(L) = 0$$

$\therefore$ $y_- = A \sin\left(\frac{\omega}{c} x\right)$ → left end

$y_+ = B \sin\left(\frac{\omega}{c}(x - L)\right)$ → rt. end.

$$y_- = y_+ \Big|_{L/2}$$

$$A \sin\left(\frac{\omega L}{2c}\right) = -B \sin\left(\frac{\omega L}{2c}\right)$$

$$A = -B$$

and

$$-M\omega^2 A \sin\left(\frac{\omega L}{2c}\right) = T\left[B\frac{\omega}{c}\cos\frac{\omega L}{2c} - A\frac{\omega}{c}\cos\left(\frac{\omega L}{2c}\right)\right]$$

$$-M\omega^2 A \sin\left(\frac{\omega L}{2c}\right) = -\frac{2T\omega}{c} A \cos\left(\frac{\omega L}{2c}\right)$$

$$\cot\left(\frac{\omega L}{2c}\right) = \frac{cL\mu}{2\omega T}\,\omega^2\,\frac{M}{\mu L}$$

$$= \frac{\cancel{c}}{c^{\cancel{2}}}\,\frac{\omega^{\cancel{2}}}{\cancel{\omega}}\,\frac{L}{2}\,\frac{M}{\mu L}$$

$$\therefore \quad \frac{2c}{\omega L}\cot\left(\frac{\omega L}{2c}\right) = \frac{M}{\mu L}.$$

higher n's $\rightarrow$ periodicity of cot.

→ Green's Function for Sturm - Liouville Problem

recall general S-L problem:

$$-\frac{d}{dx} T(x) \frac{d\rho}{dx} + V(x)\,\rho(x) = \omega^2 \sigma \rho(x)$$

with eigenfunctions $\rho_n(x)$ s/t

$$-\frac{d}{dx} T(x) \frac{d\rho_n}{dx} + V(x)\,\rho_n(x) = \omega_n^2 \sigma(x)\,\rho_n(x)$$

Now: recall Green's "function" $G(x,y)$ for linear operator $\underline{\underline{L}}(x)$

here $\underline{\underline{L}}(x) = -\frac{d}{dx} T \frac{d}{dx} + V - \omega^2 \sigma(x)$

satisfies:

$$\underline{\underline{L}}(x)\, G(x,y) = \delta(x-y)$$

n.b. $G(x,y)$ is actually a <u>distribution</u>, a la' delta function.

Point of interest of G.F. is solution of driven problem, i.e.

$$\underline{\underline{L}}(x)\,\rho(x) = f(x,\overset{\text{parameter}}{\omega}) \qquad (1)$$

then $\rho(x) = \int dy\, G(x,y)\, f(y,\omega)$

is particular solution of (1).

$\Rightarrow$ a) $G(x,y) \equiv \underline{\underline{L}}^{-1}\,\delta(x-y)$

b) as any function $f(x,\omega)$ can be written as superposition of delta functions

$$f(x,\omega) = \int f(y,\omega)\,\delta(x-y)$$

thus, any solution of inhomogeneous S-L problem can be represented as superposition of $G(x,y)$

i.e. $\rho(x) = \int dy\, \underbrace{G(x,y)}_{\text{impulse response}}\, f(y,\omega)$

i.e. $\underline{\underline{L}}^{-1}\,\delta(x-y)$

$$= \underline{\underline{L}}^{-1} f(x,\omega)$$