Stochasticity in Vehicular Traffic Flow

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1 Introduction

Traffic is often modeled as a fluid and while this description is macroscopically sound, it is perhaps unintuitive and makes it difficult to understand traffic in the context of single cars. Here we will discuss a more intuitive model of traffic flow using stochastic processes where each car is represented as a single unit and where the overall behavior is determined from principles at the single car level. In this way we can describe how cars come to move together in units we will call car clusters and how they can similarly break apart. We will consider specific examples including a circular road with no on or off ramps to gain insight into the use of the model.

2 Single Cluster Traffic Model

We can begin by considering the simple case of the dissolution of a car cluster, such as the moments immediately following the changing of a traffic light from red to green. We know intuitively that the process will strictly be one of decay, in which the first car begins to move away from the cluster, followed by the second, etc. until the cluster is completely dissolved. This situation is easily described by a master equation. If $P(n,t)$ is the probability of finding n cars in the cluster at time t, and our decay rate from is $W_{n+1,n}$ then we have:

$$\frac{\partial P(n,t)}{\partial t} = W_{n+1,n}P(n+1,t) - W_{n,n-1}P(n,t) \qquad (1)$$

But what should W be? Intuitively we know that W should not change depending on n, after all we dont accelerate faster or slower from a stop light depending on how many cars were in front of, or still are behind us. Instead we can treat W simply as a constant representing the mean reaction time of drivers, ω. If we start with n_0 cars in the cluster we can write the

set of equations:

$$\frac{\partial P(n_0, t)}{\partial t} = -\omega P(n_0, t)$$

$$\frac{\partial P(n, t)}{\partial t} = \omega[P(n+1, t) - P(n, t)] : 0 < n < n_0$$

$$\frac{\partial P(0, t)}{\partial t} = \omega P(1, t) \tag{2}$$

Solving the first equation and proceeding iteratively we find

$$P(n, t) = \frac{(\omega t)^{n_0 - n} e^{-\omega t}}{(n_0 - n)!} : 0 < n \le n_0$$

$$P(0, t) = 1 - \sum_{j=0}^{n_0 - 1} \frac{(\omega t)^j}{j!} e^{-\omega t} \tag{3}$$

This is of course the Poisson function and we can simplify our notation a bit with the expression:

$$f(k.\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{4}$$

Additionally we can compute the average value of n, $< n >$ and the variance, σ^2.

$$< n >= \sum_{n=1}^{n_0} n f(n_0 - n, \omega t) \tag{5}$$

$$\sigma^2 = \sum_{n=1}^{n_0} n^2 f^2(n_0 - n, \omega t) - < n >^2 \tag{6}$$

We can see the $< n >$ decreases linearly in time to first order by expanding the exponential in the sum so that $< n > \approx -\omega t$. We can visualize the dissipation by plotting for $P(n, t)$ a few different times as seen in Fig. 1.

The most natural extension of our model is to allow the addition of cars to the cluster. We can do this simply by changing our master equation such that:

$$\frac{\partial P(n, t)}{\partial t} = \omega_+(n-1)P(n-1, t)$$

$$+\omega_-(n+1)P(n+1, t)$$

$$-[\omega_+(n) + \omega_-(n)]P(n, t) \tag{7}$$

Additionally, let us consider a circular road with a finite number of cars. Now again we have to decide how to interpret our ω_+ and ω_- terms. We can leave ω_- as a constant since there is no real difference between a cluster
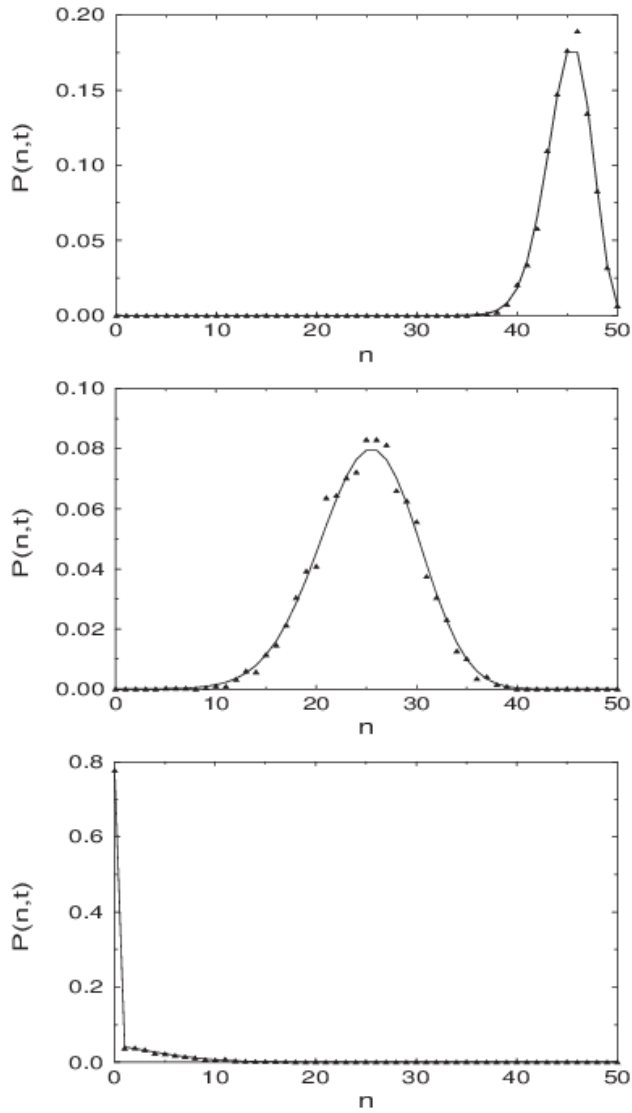
Figure 1: Plots have been generated with and $\omega t = 5, 25,$ and 55 respectively. The triangles are from a Monte Carlo simulation of 5000 trials.

of cars traveling at a constant velocity and a cluster of cars stopped at a traffic light. The situation for ω_+ on the other hand is more complicated. We know from experience that a cars ideal traveling velocity will depend on the proximity of other cars, if we are very close to the car ahead of us we instinctively slow down to allow room to brake in the case of an emergency, in the same way if there is no one ahead of us we are free to travel at our chosen maximum speed. With this motivation in mind we propose the following equation for a cars velocity:

$$v = v_{max} \frac{\Delta x^2}{D^2 + \Delta x^2} \tag{8}$$

With D being a constant and Δx being one of either $\Delta x_{free}(n)$ or Δx_{clust}. This allows us to calculate ω_+ as:

$$\omega_+(n) = \frac{v(\Delta x_{free}(n)) - v(\Delta x_{clust})}{\Delta x_{free}(n) - v(\Delta x_{clust})} : 1 \leq n < N \tag{9}$$

We know that $\Delta x_{free}(n))$ must be a function of n since the space available for cars not in the cluster will depend on how many cars are in the cluster, on the other hand we can assume that Δx_{clust} is a constant since drivers in a traffic jam do not care how many cars are in front or behind them. With these assumptions in mind we calculate the total length of a car cluster as:

$$L_{clust}(n) = ln + \Delta x_{clust} C(n) \tag{10}$$

Where l is the length of a car and $C(n) = 0$ when $n = 0$ and $C(n) = n - 1$ when $n \geq 1$ then the total length of road is given by:

$$L = ln + \Delta x_{clust} C(n) + l(N - n) + \Delta x_{free}(N - C(n)) \tag{11}$$

And we can solve for Δx_{free} as:

$$\Delta x_{free} = \frac{L - lN - \Delta x_{clust} C(n)}{N - C(n)} \tag{12}$$

The last thing we need is a transition probability that will initiate a car cluster, naturally it will increase with the total number of cars so we choose $\omega_+(0) = pN$ Where p is just a constant. Next we consider the average cluster size, $< n >$. If we take

$$\frac{\partial < n >}{\partial t} = < \omega_+(n) > - < \omega_-(n) > \tag{13}$$

and assume is reasonably large and not changing too quickly then we can approximate

$$\frac{\partial < n >}{\partial t} \approx \omega_+(< n >) - \omega_-(< n >) \tag{14}$$

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Then a stationary cluster will of course have $\frac{\partial n}{\partial t} = 0$ such that $\frac{\omega_+(<n>)}{\omega_-(<n>)}$ yielding:

$$\frac{v_{max}}{\omega_-}[\frac{\Delta x_{free}^2(<n>)}{D^2 + \Delta x_{free}} - \frac{\Delta x_{clust}^2(<n>)}{D^2 + \Delta x_{clust}^2(<n>)}]$$
$$= \Delta x_{free}(<n>) - \Delta x_{clust}(<n>) \quad (15)$$

Which has nontrivial solutions:

$$\Delta x_{free} = \frac{D}{2\alpha}\left[\frac{v_{max}D}{\omega_-} \pm \sqrt{(\frac{v_{max}D}{\omega_-})^2 + 4\frac{v_{max}\Delta x_{clust}(<n>)\alpha}{\omega_-} - 4\alpha^2}\right] \quad (16)$$

where $\alpha = D^2 + \Delta x_{clust}^2(<n>)$.

Now if we stick with the assumption that N is large and treat $C(n) \approx n$ we find

$$<n> = \frac{N(\Delta x_{free} + l) - L}{\Delta x_{free} - \Delta x_{clust}} \quad (17)$$

Where we see we have two solutions for $<n>$. One stable, with Δx_{free}^+ and one unstable, with Δx_{free}^-. In the special case where there is no bumper to bumper distance, $\Delta x_{clust} = 0$ we obtain the result:

$$\Delta x_{free} = \beta \pm \sqrt{\beta^2 - D^2} \quad (18)$$
$$<n> = N + \frac{l - L}{\beta \pm \sqrt{\beta^2 - D^2}} \quad (19)$$

with $\beta = \frac{v_{max}}{2\omega_-}$.

3   Multi-cluster Traffic Model

One can also obtain a reasonable model of traffic flow that includes the probability that multiple clusters form along the road. A few simplifications make this problem tractable. As in the single cluster model, one can assume the rate at which cars leave the cluster is fixed. We also restrict ourselves to the one-lane circular road of length L with no on/off ramps. The relevant states of the system can be labeled by the $N + 1$-dimensional vector

$$\mathbf{N}(t) = (N_0(t), N_1(t), N_2(t), ..., N_N(t)), \quad (20)$$

where N is the total number of vehicles and and N_k is the number of size-k clusters. There can only be one cluster of size N and if there are no clusters, the first component is N, i.e. there are N clusters of size zero. In the previous model, the number of clusters was restricted to 1 total. This fixed the cluster vector to have a one to one correspondence with the

number of cars in the cluster. We see now that the situation is a bit more complicated given that for n cars in a jam, there are many states that satisfy that condition. The length of the road is chosen to be some multiple of the space a vehicle occupies when it is in a cluster. That is,

$$L = M(l + \Delta x_{\text{clust}}), \tag{21}$$

where M is some natural number such that $N \leq M$. The velocity of a given car depends on the local density or equivalently on the distance (headway) between its front bumper and the rear bumper of the next vehicle. In general, the front bumper to front bumper distance is given by $l + \Delta x$. The total number of jam clusters can be written,

$$N_{\text{cl}} = \sum_{k=1}^{N} N_k \tag{22}$$

and the total number of jammed cars is

$$n_{\text{jam}} = \sum_{k=1}^{N} k N_k \tag{23}$$

so the total number of free cars is,

$$n_{\text{free}} = N - n_{\text{jam}}. \tag{24}$$

The length of a congested cluster of size k is $lk + (k-1)\Delta x_{\text{clust}}$ and so the total length of congested cars is $L_{\text{clust}} = ln_{\text{jam}} + (n_{\text{jam}} - N_{\text{cl}})$. The average distance between cars not in a jam is given by

$$\Delta y_{\text{free}}(n_{\text{jam}}, N_{\text{cl}}) = \frac{M - N + (M - n_{\text{jam}} + N_{\text{cl}})\Delta y_{\text{clust}}}{N - n_{\text{jam}} + N_{\text{cl}}} \tag{25}$$

where $l\Delta y = \Delta x$.

In order to construct the master equation for this system, we must develop an ansatz for $\omega_+(n_{\text{jam}}, N_{\text{cl}})$, the transition frequency for attaching a car to any of the N_{cl} clusters. The transition frequency for detachment is simply $\omega_-(n_{\text{jam}}, N_{\text{cl}})$ which we still assume is constant. A reasonable form for the rate of attachment is, in the case that the free cars are distributed uniformly so the headway for each one is given by the mean headway $\Delta y_{\text{free}}(n_{\text{jam}}, N_{\text{cl}})$, is a modified version for of the single cluster traffic jam. For multi-cluster traffic, the condition $n_{\text{free}} < N_{\text{cl}}$ is possible and so a correction term is added to account for the fact that there are no cars in between some of the N_{cl} clusters. The corrected attachment frequency is given by

$$\omega_+ = \frac{b}{\tau} \frac{\omega_{\text{opt}}(\Delta y_{\text{free}}(n_{\text{jam}}, N_{\text{cl}})) - \omega_{\text{opt}}(\Delta y_{\text{clust}})}{\Delta y_{\text{free}}(n_{\text{jam}}, N_{\text{cl}}) - \Delta y_{\text{clust}}} \times R(n_{\text{jam}}, N_{\text{cl}}) \tag{26}$$

$$\omega_- = \frac{1}{\tau} = \text{const} \tag{27}$$

where $b = \frac{v_{\max}\tau}{l}$ and

$$R(n_{\text{jam}}, N_{\text{cl}}) = 1 + (\frac{n_{\text{free}}}{N_{\text{cl}}} - 1)\theta(N_{\text{cl}} - n_{\text{free}}) \tag{28}$$

The step function has been used to turn on the change in the rate when the condition $n_{\text{free}} < N_{\text{cl}}$ is met. Roughly speaking, it reduces the probability rate by a factor proportional to the probability there exists at least one free car in between the clusters. Further, this constraint fixes $\omega_+(N, N_{\text{cl}}) = 0$ so the number of congested cars satisfies $n_{\text{jam}} \leq N$.

We can write the stochastic variables as an $N-$dimensional vector $\mathbf{N} = (N_1, N_2, N_3, ..., N_N)$. The master equation may then be written as,

$$\frac{1}{\tau}\frac{dP(\mathbf{q}, T)}{dT} = (N - n + 1)\omega *_+ P(\mathbf{q} - \mathbf{q}_1, T)$$

$$+ \sum_{k=2}^{N} (N_{k-1} + 1)\omega_+(n_{\text{jam}} - 1, N_{\text{cl}})P(\mathbf{q} + \mathbf{q}_{k-1} - \mathbf{q}_k, T)$$

$$+ (N_1 + 1)\omega_-(n + 1, N_{\text{cl}} + 1)P(\mathbf{q} + \mathbf{q}_1, T)$$

$$+ \sum_{k=1}^{N-1} (N_{k+1} + 1)\omega_-(n_{\text{jam}} + 1, N_{\text{cl}})P(\mathbf{q} + \mathbf{q}_{k+1} - \mathbf{q}_k, T)$$

$$- [(N - n_{\text{jam}})\omega *_+ + N_{\text{cl}}(\omega_+(n_{\text{jam}}, N_{\text{cl}})$$

$$+ \omega_-(n_{\text{jam}}, N_{\text{cl}}))]P(\mathbf{q}, T) \tag{29}$$

where T is time measured in units of the time constant τ, and $\mathbf{q} = \sum_k N_k \delta_{i,k}$. The processes that have been considered by this model include condensation and evaporation of car clusters caused by stochastic one-step car attachments or detachments processes. The stationary distribution is reached for $P(\mathbf{q}) = lim_{T\to\infty}P(\mathbf{q}, T)$. The master equation gives this condition by setting the time derivative of the probability distribution to zero, i.e.

$$\frac{dP(\mathbf{q}, T)}{dT} = 0 \tag{30}$$

We focus on the stationary distribution for clusters in the thermodynamic limit of an infinitely large system such that, $M \to \infty$ and $N \to \infty$ and the average quantity $C(k) = <\frac{N_k}{M}>$ is the density of clusters of size k given by

$$C(k) = M^{-1}\sum_k iP_k(i) = \frac{N_k^*}{M} \tag{31}$$

where $N*_k$ is the average value of the number of clusters of size k. The fluctuations in the thermodynamic limit are negligible and therefore we can write down $P_k(i) = M^{-1}\delta(\frac{i}{M} - \frac{N_k^*}{M})$ which is the stationary probability

distribution that there are i clusters of size k. In general the function $P_k(i)$ can be obtained by summing over all values of N_m in $\mathbf{q} = \sum_m N_m \mathbf{q}_m$ except for the cluster size in question, $N_k = i$. The time dependence of the clusters densities

are then given by the following coupled equations

$$\frac{C(0)}{dT} = -pC(0) + C(1) = 0$$
$$\frac{C(k)}{dT} = [Q + \delta_{k,1}(p - Q)]C(k-1)$$
$$-(1+Q)C(k) + C(k+1) = 0 : k \geq 1 \tag{32}$$

with $C(0) = \frac{n_{\text{free}}}{M}$, $p = \tau\omega_+^*$, and $Q = \tau\omega_+(n_{\text{jam}}, N_{\text{cl}})$. The stochastic perturbation p is a parameter that gives the probability rate that a car decelerates into the congested phase. One can then obtain the following relation between cluster densities:

$$C(k) = C(0)pQ^{k-1} : k \geq 1 \tag{33}$$

The physical solution must be taken to be $Q < 1$ or else the total number of clusters and congested cars, N_{cl} and n_{jam} diverge. At the end of the day, in the stationary limit $T \to \infty$ the following equations hold from the master equation;

$$C(0) = \frac{c}{c_{clust}} \frac{\gamma^2}{p + \gamma^2} \tag{34}$$
$$r = \frac{p}{p + \gamma^2} \tag{35}$$
$$s = \frac{1}{\gamma} \tag{36}$$

where $r = \frac{n_{\text{jam}}}{N}$ is the fraction of congested cars and $s = \frac{rN}{N_{\text{cl}}}$ is the average cluster size and $\gamma = 1 - Q$. From these equations one can obtain the expression $\frac{n_{\text{free}}}{N_{\text{cl}}} = \frac{1-Q}{p}$. So for small enough values of p, the condition that $n_{\text{free}} > N_{\text{cl}}$ is met and $R = 1$ for the rate ω_+. One can numerically solve these set of equations which is is considered the solution in the thermodynamic limit as all the variables are now written as ratios and don't include N or M on its own. We are interested in the limit that $p \to 0$ because rarely in actual traffic will a driver randomly decelerate.

One can obtain critical densities by setting $Q = 1$ and solving for the critical values of $\Delta y_{1,2}$ such that

$$\Delta y_{1,2} = B \pm \sqrt{B^2 - d^2 + 2B\Delta y_{\text{clust}}} \tag{37}$$

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and

$$c_{1,2} = \frac{1}{1 + \Delta y_{1,2}} \tag{38}$$

where

$$B = \frac{bd^2}{2[d^2 + (\Delta y_{\text{clust}})^2]} \tag{39}$$

The corresponding asymptotic solution exists in the range $c_1 < c < c_{clust}$ where large clusters coexist with regions of free traffic. Here, c_1 is the density where homogeneous traffic flow starts to become unstable and large clusters begin to form as c is increased. For $c < c_1$ there exists only one homogeneous traffic solution and it also holds in the case that $c_2 < c_{\text{clust}}$ in the region $c_2 < c < c_{\text{clust}}$. However the region $c_2 < c < c_{\text{clust}}$ also has an extra solution suggesting that there exists some sort of jump between the two possible states. What is left to determine is which state represents the actual stable state of the system.

Monte Carlo simulations with experimental parameters matching with data obtained from German highways show 3 distinct traffic phases. A free flow phase for $c < c_1$, a congested phase at $c_1 < c < c_{\text{jump}}$, and a highly viscous homogeneous state $c_{\text{jump}} < c < c_{\text{clust}}$. Some results of the MC simulations are shown in Fig. 2. MC simulations have also been done to show the jump from one metastable state to the other as shown in Fig. 3. The two states correspond to the dense homogeneous state and the cluster phase.

4 Discrete Time and Space Traffic Models

4.1 Cellular Automata

Another approach used to study traffic phenomena is the simulation of traffic flow via cellular automata. In this case, both space and time are considered discrete. There exists a lattice of sites which can be either considered on or off. The time discretization comes in the form of a well defined update process which follows specific rules at each time step $t \to t + 1$. The rules are motivated by empirical observatioins of traffic flow. Different models for traffic flow will have different rules for the update process. Because space and time is discrete, so to are the velocities and thus the states of the interacting units are quantized. The finiteness of the number of states arises in traffic models as a limit on the possible velocity of an interacting unit set by the speed limit.

A simple and popular model for traffic flow is called the Nagel-Schreckenberg (NS) model from which variations are made to study the effects of slightly different updates processes. The simplest example is that of a one lane circular highway. For the NS model, the speed of each vehicle can take $v_{max} + 1$ integer values in the range of $0, 1, 2, ..., v_{max}$. The headway for the
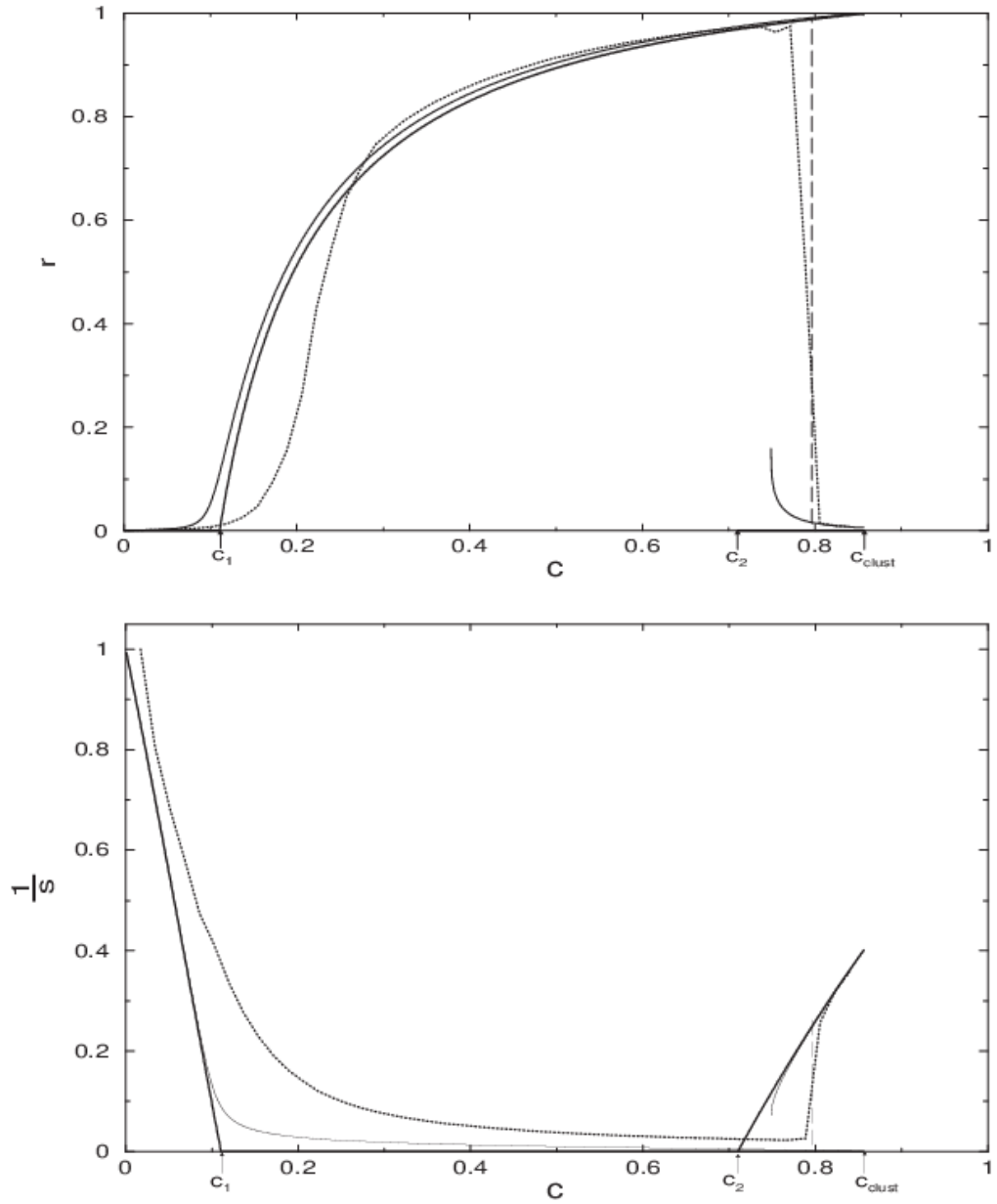
Figure 2: The relative part of the congested cars r (upper) and $\frac{1}{s}$ (lower) vs the dimensionless density c for different values of M. Thin solid line: $p = 0.001$, $M = \infty$; thick solid line: $p = 0$, $M = \infty$; dotted line: MC simulation for $p = 0.001$, $M = 50$. The vertical line represents c_{jump}
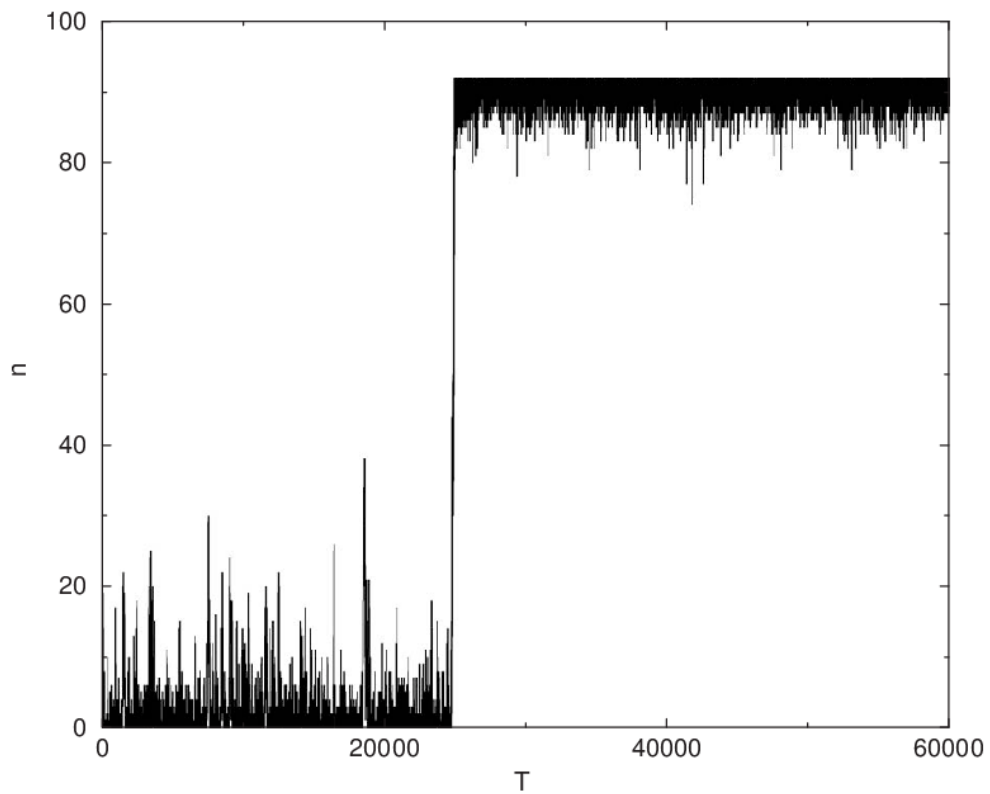
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Figure 3: A specific stochastic trajectory showing the switching from the homogeneous state to the cluster phase. In this simulation M = 100 and N = 92 and p = 0.001. The vertical axis is the number of congested cars.

nth vehicle is the distance from it and the $n + 1$th vehicle at time step t is $d_n = x_{n+1} - x_n$. The update rules are motivated in such a way to avoid collisions and include some element of stochasticity while also including the general tendency of the driver to reach the maximum speed limit. Each vehicle is updated in parallel, or said another way, what the driver decides to do is assumed to only depend on the vehicle ahead of it at given time t. The rules are as follows:

Step 1: Acceleration to speed limit. If $v_n < v_{max}$, the speed of the nth vehicle is increased by one, but v_n stays the same if $v_n = v_{max}$. This can be contained in the following expression

$$v_n \to min(v_n + 1, v_{max}).$$

Step 2: Avoiding a crash/ If $d_n \leq v_n$, the speed is reduced to $d_n - 1$. This can be put in the form,

$$v_n \to min(v_n, d_n - 1).$$

Step 3. Stochasticity. If $v_n > 0$, the speed of the nth vehicle is reduced by 1 with probability p but v_n does not change if $v_n = 0$.

$$v_n \to max(v_n - 1, 0)$$

with probability p.

Step 4: Update Vehicles in Parallel. Each vehicle will be moved according to its new velocity from steps 1 through 3.

$$x_n \to x_n + v_n.$$

This simple model is able to reproduce some essential features of real traffic flow, one of which is the spontaneous generation of traffic jams as can be seen in Fig. 4. Also, the darkest regions representing the traffic jams are density waves which move backward. As the density decreases, however, the wave propogates forward as can be seen in the simulation.

4.2   Approximate Mean Field Approaches

To obtain approximate analytical results for cars on a discrete spacetime lattice, one can use the methods of mean-field theory. For traffic flow, several variations of the mean-field approach exist; some of them are: site-oriented naive mean-field (SOMF) theory, paradisical mean-field theory, car oriented mean-field (COMF) theory, and cluster approximations. All of the aforementioned methods yield exact results in the case of $v_{max} = 1$ except for SOMF theory.

For the site-oriented theories, each site has a discrete set of possible values and the system is of size L with periodic boundary conditions. Each
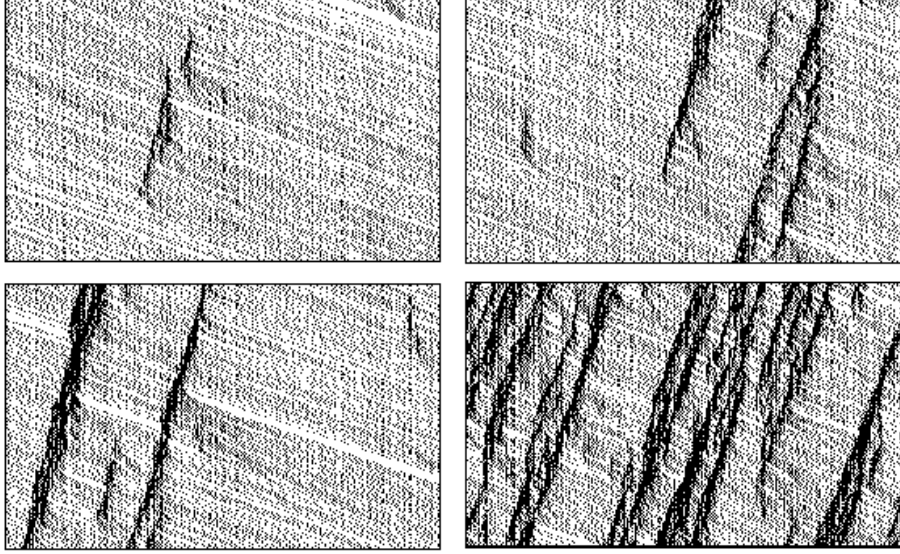
Figure 4: A cellular automata simulation of one-lane circular traffic flow using the NS model. Results for different car densities are shown. The time axis is downward.

site is allowed to take $v_{max} + 2$ possible values corresponding to the empty cell state, and the occupied cell state for all $v_{max} + 1$ possible velocities.

The SOMF theory can be applied to the NS model of traffic flow by expressing the probability $c_v(i; t + 1)$ that there exists at time $t + 1$, a vehicle with velocity v in the ith cell to the probabilities of the previous time step. Summing all of these probabilities gives the probability that a site i is occupied at time t. That is, $c(i; t) = \sum_{v=0}^{v-max} c_v(i; t)$. One can then obtain the following set of equations in the case that $v_{max} = 1$:

$$c_0(i; t + 1) = c(i; t)c(i + 1; t) + pc(i; t)d(i + 1; t) \tag{40}$$

$$c_1(i; t + 1) = qc(i - 1; t)d(i; t) \tag{41}$$

with $d(i; t) = 1 - c(i; t)$ is the probability that there is no car at site i at time t. The probability $p = 1 - q$ is that for any vehicle with nonzero velocity to decelerate at the randomization step. The expression for $c_0(i; t+1)$ contains both situations where the velocity could be zero at the next time step. This could either be due to there being a vehicle ahead of it at $i + 1$ or it could have randomnly decelerated even if there were no vehicle ahead of it. The second equation says that the velocity could be nonzero at $t + 1$ if there wasn't a vehicle at time t at i and if there was a vehicle at time t at $i - 1$ that did not accelerate which has the probability q.

Translation invariance of the stationary state solution exists due to the use of periodic boundary conditions, so the final stationary state solution of

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$c_v(i;t)$ should not depend on i.For this model the flux is then given by c_1 using the second equation we have,

$$J = c_1 = qc(1 - c).$$ (42)

The SOMF theory underestimates the flux J compared to the exact expression from the deterministic NS model where $p = 0$ and $v_{max} = 1$ which is,

$$J = min(c, 1 - c).$$ (43)

The exact expression is linear as it rises to the peak at $\frac{1}{2}$ and goes back down linearly to zero when $c = 1$. The SOMF theory gives a quadratic function of c which rises with negative curvature and peaks at $\frac{1}{4}$ and goes back down to zero when $c = 1$.

The discrepancy is due to the SOMF theory including states which do not exist in the steady state deterministic NS-model with parallel updating. These states are called "Garden of Eden" states or paradisical states because one cannot get back to the state once one has left. One example of such a state is where two cars are right next to one another yet the one that is ahead of the other has speed $v + 1$ and the one behind has speed v. Such a state in the deterministic model could have only come from a state that the NS model could not include. It means that at the time step earlier, they were both at the same location. The improved SOMF theory is called pMF or paradisical mean field theory where one excludes such states. The pMF theory gives back the exact flux of the NS model for $v_{max} = 1$.

5 Bibliography

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