Mathematical Valuation of Financial Derivatives

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1 Introduction

Ever since the global financial crash of 2008, the mathematical valuation of financial derivatives has been thrust into the public eye. Many who call themselves practitioners of the Black Scholes model in finance are relatively unaware of its basic mathematics, and were taken fully by surprise by the events of 2008. A closer scientific evaluation of its derivation, and especially its sometimes tenuous and often simply incorrect assumptions, left the careful examiner unsurprised. The belief that Black Scholes is a "black box" that ensures everything will always run smoothly was, and still is, economically dangerous. Apart from an eyes wide shut approach leading to an economic climate of the blind leading the blind, upon serious examination of market data we see that there exists not only a possibility that this black box will fail us, but a probabilistic likelihood of failure on a certain time scale (likely quite small). An understanding of why Black Scholes seems to work so well in a certain domain, and also where it seems to fail, is incredibly important going forward. In the following paper, we lay out a background in finance intended for a scientific audience. After that, we derive Black Scholes and apply it to simple European call options. The remainder of the paper is meant to highlight the failings of Black Scholes, and attempts at fixing these to create a model that closer resembles the real world.

2 Background Finance

In order to understand how the mathematics is applied to economics, a basic understanding of finance is required. Here we will very briefly discuss the required financial concepts and definitions (for a more complete discussion, see Neftci's book [7]). The broadest concept is that of financial derivatives, which are the focus of all quantitative economic models discussed in this paper. A derivative is generically defined as any contract which derives its value from financial instruments such as stocks, bonds, currencies, and commodities. To understand this definition, it is easiest to look at some examples of common derivatives used in markets. For example, sometimes it is in the interest of a business to know in advance the exact price of a certain commodity they will need at some future time. In order to bypass the unknown fluctuations in the

prices of the commodity from present to future, they may desire to pay some set amount for this commodity now and receive a derivative contract promising them the commodity at a future time. This specific kind of contract is either known as a forward or future contract, depending on how and when the funds are payed out. It can be used to capitalize on market speculation, by entering this contract if you think the underlying will rise dramatically, or for hedging, if you want to avoid worrying about market fluctuations altogether. These contracts themselves are also traded on the market, just like a stock or commodity, by people with no interest in the underyling asset whatsoever.

The problem with forwards and futures is that the holder of the contract has to pay the originally set price for the commodity no matter the price of the underlying asset at maturity. A second type of derivative, called a European call option, takes away this obligation to make the trade at maturity. The contract still gives the holder the right to buy the underlying at maturity for the agreed upon price, called the strike price, but if the market price is lower than the strike price the holder does not have to exercise the option. If this is the case, the holder would not pay the strike, would not receive the underlying, and would either buy it on the open market for cheaper or not buy at all. A put option is the exact opposite, giving the holder the right to sell an asset for some amount at maturity. Because the risk of the market is transferred entirely to the seller of these contracts, they are compensated by charging the holder some premium cost for the option itself. The fair price, or value, of this premium cost at any given time is a fundamental question that quantitative finance tries to answer with stochastic methods, akin in importance to the equation of motion in classical physics. While the term fair may seem subjective, it has a strict economic definition that guides the derivation of pricing models. This definition has to do with the concept of arbitrage, which is the opportunity to make a totally risk-free investment which guarantees a profit above that dictated by the interest rate. This is clearly not sustainable for everyone in the market, and according to economic theory the market will work to eliminate all arbitrage opportunities if they should arise. The idea of a fair price, then, is one which excludes any arbitrage opportunities.

3   Derivation of Black Scholes

The difficulty in finding the fair value of an option occurs because of the inherently random nature of the value of the underlying asset itself. It therefore seems reasonable to utilize stochastic methods to come to some solution here. In fact, the equations of Brownian motion were first derived for use in pricing stock shares before Einstein's famous derivation, in a doctoral thesis by Bachelier [2]. This work used the idea of a relative share price as the stochastic Brownian variable, and Bachelier deduced a Gaussian form for its probability distribution. This was problematic for the reason that it implies a finite probability that the stock price can become negative, and it was for this reason that some 60 years later Samuelson proposed to consider instead a quantity called the raw

return of the share price, defined as $r = (S_{i+1} - S_i)/S_i$. Here S stands for the share price, and the i'th index denotes a discrete time step. Let us go one step further and define a return as

$$R \equiv \log{(1 + r)} = \log{(S_{i+1})} - \log{(S_i)}. \tag{1}$$

We see that $\log{(1 + r)} \approx r$ in the limit $r \ll 1$, namely for small raw returns these two definitions are equal. This definition of a return is especially convenient because it is distributed normally if we are to assume that share prices are distributed lognormally. The assumption of a lognormal distribution of share prices is much more natural for markets than Bachelier's normal distribution, as it excludes the possibility of negative pricing while still allowing a Brownian motion description. In this description we see that it is actually the logarithm of the share price which undergoes the Brownian motion, giving an exponentiated Brownian process. The corresponding lognormal random walk process is then called Geometric Brownian motion, and is given by

$$dS = \mu S dt + \sigma S dX. \tag{2}$$

Here dX is a random Brownian increment, S is the share price, μ is the drift term, and σ is the volatility. The drift term is necessary such that the mean price continues to increase over time due to, for example, interest. The volatility is a measure of the fluctuations a stock experiences, and is notoriously difficult to predict at any given time.

In equation 2, we have a stochastic differential equation for some lognormal random walk process which is assumed to accurately model the value of our underlying asset. This can now be used to help come to some fair price for an option who derives its entire value from this asset. Following the derivation by Fisher Black and Myron Scholes [4], we write the option value as $V(S, t)$. By applying Ito's Lemma to equation 2, we can write the differential option value as

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt. \tag{3}$$

We will use this relation in a particular portfolio (whose elements are a combination of financial instruments and derivatives) to find the no arbitrage fair price of any option. In this portfolio we can either hold the element "long" or "short", relating to having a positive or negative amount of the element in question respectively. Holding a quantity long is intuitive, simply denoting possesion, but holding a quantity short is less straightforward. Being short in this sense means selling the quantity without ever actually possesing it first, with the idea of acquiring it in the necessary time to deliver it when required.

Now imagine puting together a portfolio of a particular option held long and an amount, Δ, of the underlying held short. The value of this portfolio is then $\chi = V(S, t) - \Delta S$. Assuming constant Δ, and utilizing equation 3, the differential portfolio value is given by

$$d\chi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS. \tag{4}$$

This equation is seen to rely on both deterministic terms and terms related to the stochasticity of the underlying. If it is possible to eliminate the stochastic terms entirely, risk would be entirely eliminated and we could say something definitive about this portfolio. This can be done if both stochastic terms in dS were set to cancel each other out. If the amount of underlying in this portfolio is set to the special value of

$$\Delta = \frac{\partial V}{\partial S}, \tag{5}$$

then the randomness of the portfolio value vanishes. This is called delta hedging, and in practice it is difficult due to the time varying nature of the quantity $\partial V / \partial S$ itself.

Once this portfolio is set up and the appropriate Δ value calculated, we see that the differential portfolio value is

$$d\chi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{6}$$

We have now constructed an entirely risk free porfolio, and all that is left is to enforce the principle of no arbitrage. This principle implies that a risk free portfolio such as the above must be exactly equal to the amount we would get by putting the same amount of cash in a risk-free interest account. For a given interest rate r, this is known to be given by

$$d\chi = r\chi dt. \tag{7}$$

If the growth were any more than the interest rate we could borrow from a bank with the set interest rate r and invest entirely in this portfolio, making a risk-free profit. If it were less than the interest rate, we could hold the option short instead, delta hedge it, and invest the earnings in the bank. Both of these scenarios represents arbitrage, so the fair porfolio value must explicity satisfy equation 7. Substituting equations 5 and 6 into equation 7, and keeping in mind that $\chi = V(S, t) - \Delta S$, we get the famous Black-Scholes equation

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}. \tag{8}$$

4   Simple Example of Black Scholes Application

This equation is a linear parabolic partial differential equation, which is known to be simple to solve numerically. Inspecting the equation shows that it is quite general, and does not exhibit any information about the type of option we are attempting to value in the first place. This information is put in when solving the Black Scholes equation in the form of a final condition (as opposed to the more common initial conditions). This is because the only time we actually know the value of the option in the real world is at maturity.

While more complicated contracts don't have analytical solutions, and are solved numerically using Black Scholes, it is still insightful to examine the most

basic solution for call options. This solution can be arrived at using the appropriate Green's function, or using Fourier or Laplace transforms, but perhaps a more direct method in this context is by again using stochastic analysis [8]. The Black Scholes equation can be seen to be of the form of a backward Fokker-Planck equation with $A = rS$ and $B = \sigma^2 S^2$. If we assume a constant volatility and define a conditional probability of $P(X, T|S, t)$, we know it must satisfy the analogous backward Fokker-Planck equation

$$\begin{cases} \frac{\partial P(X,T|S,t)}{\partial t} = -rS\frac{\partial P(X,T|S,t)}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(X,T|S,t)}{\partial S^2} \\ \\ P(X,T|S,T) = \delta(X-S). \end{cases}$$

The stochastic differential equation corresponding to this equation can be shown to be given by a lognormal random walk in the X variable. Defining $y = \log X$ and utilizing Ito calculus, the conditional probability for the variable y has the form

$$p(\tilde{y}, T|y, t) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[\frac{(\tilde{y} - y - (r - \frac{1}{2}\sigma^2)(T-t))^2}{-2(T-t)\sigma^2}\right] \tag{9}$$

The solution of the Black Scholes equation can then be written as

$$V(S, t) = e^{r(t-T)} \int d\tilde{y} \ p(\tilde{y}, T|y, t) V(e^{\tilde{y}}, T) \tag{10}$$

with the conditional probability defined as in equation 9.

To give an explicit solution in terms of any particular contract, we must input the relevant final condition into equation 10. For a simple call option with strike price E we have $V(X, T) = (X - E) \cdot \Theta(X - E)$. Plugging this into equation 10 and integrating over the bounds set by the Heaviside function, we can get the solution in terms of the cumulative Gaussian function

$$N(z) \equiv \int_{-\infty}^{z} \exp\left(-\frac{1}{2}x^2\right) dx \tag{11}$$

as

$$V(S, t) = SN(d_1) - Ee^{r(t-T)}N(d_2), \tag{12}$$

$$d_1 = \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{13}$$

$$d_2 = \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{14}$$

This is called the Black-Scholes option pricing formula, and upon its introduction in the 1960's led to a drastic increase in options trading. This formula, or slight variations of it, is still widely used today in options markets worldwide.

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5 Critical Evaluation of Black Scholes Assumptions

Now that the Black-Scholes model has been introduced, it becomes instructive to look at how well it actually relates to physical markets. To do this, lets first examine it's key assumptions. Upon inspection of equation 12, we see that the fair price of the derivative as provided by the Black-Scholes formula depends on the volatility, σ, which is a measure of the size of flucuations. In order to arrive at a price, then, the volatility of an underlying must be assumed. This is a hard quantity to decide upon, because its dependence on past data can be tenuous. Even more worrying is the need for volatility to be constant in our model, which is highly unlikely. An incredibly large number of factors effect the volatility of a stock, including seasonality, economic upswings, etc. Another paramater that we must preset is the risk-free interest rate r. Again, in practice this can be difficult to do because interest rate is essentially stochastic itself. Various methods exist to deal with this problem, as outlined in Wilmott [6].

While the previous required assumptions are a bit worrying, they are all harmless enough and exist more as difficulties than fundamental causes for concern. It is now time to discuss the more essential problems. A serious problem that quickly becomes apparent in looking over our derivation is that we absolutely required continuous hedging, as seen in equation 5. In practice this is impossible, and therefore what really occurs in markets is discrete hedging. There is, of course, some hedging error in the real world then, but this is often ignored because it is assumed to average out to zero. This turns out to be a reasonable assumption, as discussed mathematically in Wilmott [6], but only if the returns are really distributed as assumed. This brings us to an even more fundamental, and worrying, assumption of the model, namely that the underlying is distributed lognormally. To begin, if this is not true the hedging error will not always average out to zero and will likely be compounded. On a more basic level, if the model is indeed not distributed lognormally then the entirety of the model is invalid. Its derivation relied upon using Ito calculus assuming a Gaussian noise term, but with this noise term distributed in some non-Gaussian manner, Ito's Lemma itself must be generalized. It turns out that this is very difficult, and often wildly increases the complexity of the problem. Unfortunately, it turns out that this particular assumption is contradictory to lots of market data. One particular set of real-world data is shown in Fig 1, with the distribution of a particular underlying plotted and fit to various curves.

From this data it can be seen that these returns do not in fact follow a normal distribution, but have a much heavier tail. It turns out that a collection of such data exists and comes to the same conclusion, implying that in reality returns do not fit a Gaussian distribution for large positive or negative values. This means that actual financial markets seem to have sudden jumps in returns, perhaps an unexpected crash, more often than would be expected from a normal distribution. This may seem like a small difference, but these jumps can have serious economic consequences not accounted for by our model.
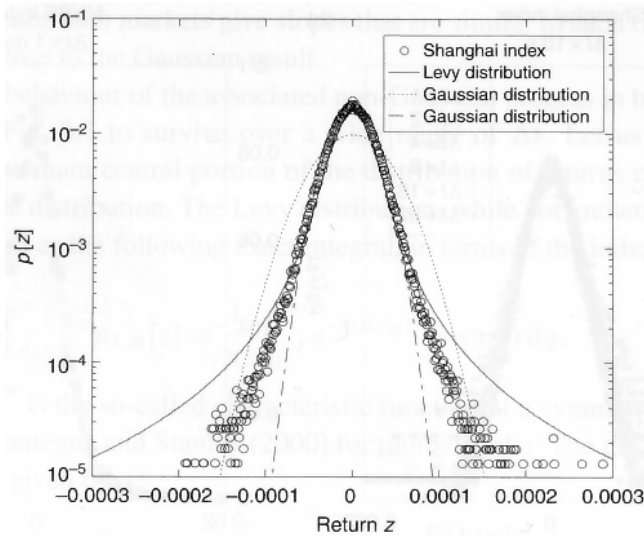
Figure 1: A comparison of market data to various distributions [5]

6 Beyond Black Scholes

In order to minimize the likelihood of a breakdown of our economic model, we must make adjustments to ensure a closer fit to market data. The most straight-forward solution would be to simply assume some non-Gaussian distribution of returns and rederive our model. This, however, presents undue complications and will be discussed later in the paper. For now, we will forgo scrapping our model entirely and simply examine small fixes. If we are to keep the Black Scholes model, one fix could be to re-examine the assumption of constant volatility. This parameter was clearly the hardest to identify, and in fact seems reasonable to model itself as a stochastic variable. We would then have the same lognormal random walk for the share price, but now also

$$d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dX_2 \tag{15}$$

with dX_2 as a second Brownian increment. The choice of functions for p and q becomes important, and many models have been proposed. The most common seem to be Ohrnstein-Uhlenbeck processes, or a similar mean-reverting process. We could now rederive a Black Scholes equation using this new assumption, but must use a different portfolio. Namely, we now how to hedge away two separate sources of randomness and so would choose a portfolio such as

$$\chi = V - \Delta S - \Delta_1 V_1. \tag{16}$$

Finding the differential porfolio value and enforcing no arbitrage is possible, but it turns out we end up with one equation in two unknowns. A new function

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called the market price of volatility risk can be added to deal with this, but it is not entirely enlightening either [6].

7   Jump Diffusion Model

We found that assuming stochastic volatility would perhaps improve our model's fit to data, but didn't give us anything entirely insightful to work with. A more drastic change would be to actually allow the model to reflect the concept that assets can jump in value. To do this, we factor in the Poisson process to our original model which already had Brownian motion accounted for. The Poisson process is defined as:

$$dq = 0 \text{ with probability } 1 - \lambda dt, \text{ or } dq = 1 \text{ with probability } \lambda dt \qquad (17)$$

The parameter λ is known as the intensity for the Poisson process. If we say that the variable S can jump to JS according to this Poisson process, we can model our logarithmic random walk as:

$$d(\log S) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW + (\log J)dq \qquad (18)$$

Variable J is a quantity distributed according to $P(J)$ and can be generalized to be random. This logarithmic distribution is essentially Itô with jump-diffusion factored in.

For the Jump-Diffusion model to be useful in finance, we want to be able to hedge while there are jumps in the assets. One option is to hedge similar to how we hedge in Black-Scholes, where we choose $\Delta = \partial V/\partial S$ to eliminate risk. This works when we have no jumps and $dq = 0$. However, we see if there is a jump $(dq = 1)$, then there is an amount that cannot be hedged. So we explore the option of hedging by picking Δ such that we minimize the variance of the portfolio. Again, we see that there is no perfect hedging in this case. We are left with two options in the jump-diffusion model; we can either hedge the diffusion similar to Black-Scholes and take on risk due to not accounting for jumps, or we can hedge the jumps themselves, and accept that our hedges will not always work. Both methods of hedging assume significant amounts of risk, which make them difficult to use if one wants to have a useful model of finance. We are essentially left with the choice of either choosing an inaccurate model in Black-Scholes or moving toward a more accurate model which increases the risk in hedging. In either case, we see that the jump-diffusion model does better reflect the real world market data.

8   Levy Processes and Pareto-Levy Distribution

We see from Black-Scholes that a Gaussian distribution for the Wiener Process does not accurately represent the distribution of real world market data. The heavy tails on the return data seem to be better represented by a Pareto-Levy

Process. It is now time to look at solving this problem at the source, namely we want to look at the general Levy Process and see which conditions can let us account for the heavy tails. Because we want our probability to be homogeneous in time and probability space, the motivation for the Levy Process arrives from the differential Chapman-Kolmogorov Equation [19]

$$\partial_t p(z,t) = -a\partial_z p(z,t) + \frac{1}{2}\sigma^2 \partial_z^2 p(z,t) + \int duw(u)[p(z-u,t) - p(z,t)] \quad (19)$$

A solution can be realized if we use the Fourier transform for the probability [20].

$$\phi(s,t) = \int_{-\infty}^{+\infty} dz e^{isz} p(z|0,t) = \exp[(ias - \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{+\infty} du(e^{isu} - 1)w(u)t)] \quad (20)$$

Here $w(u)$ is the intensity, and the variable u is the mean time between jumps. The conditions for the Pareto-Levy Process are outlined in Equations 21 and 22.

$$a = \sigma = 0, \quad (21)$$

$$w(u) = A|u|^{-\alpha-1} \text{ with } 0 < \alpha < 2 \quad (22)$$

We can ultimately reduce this characteristic function to one of a Paretian process as seen in Equations 23 and 24.

$$\phi(s,t) = \exp[-|s|^\alpha t\gamma(1 + i\beta\frac{s}{|s|}tan(\frac{\pi\alpha}{2}))] \quad (23)$$

$$\phi(s,t) = \int due^{isu}\text{Par}(\alpha,\beta,\gamma t; u) \quad (24)$$

Here we see that the characteristic equation is simply the Fourier transform of the Paretian conditional probability $\text{Par}(\alpha, \beta, \gamma t; u)$. Using these conditions for our Levy Process is useful because real world market data appears to line up much better with this adjusted distribution. The one downside is that the Pareto-Levy process has infinite variance in returns, whereas the market data does not.

At the moment, Black-Scholes is the preferred model due to its usefulness in finance, however the Pareto-Levy Process is still being developed as this model seems to be the best option at modeling real world market data more accurately and more directly. There is still of plenty debate as to how one should develop mathematical finance models, and which ones are best, but in developing said models it is vital to keep in mind that they are being created to fit observed market data, and not the other way around.

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