

Physics 161: Black Holes: Lecture 14 and 15: 21 Feb -26 Feb 2013

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14 Rotating Black Holes: the Kerr Metric

14.1 Kerr Metric

Almost all stars rotate. Thus when massive stars collapse into black holes at the end of their lives the resulting black holes should also be rotating. Thus the Schwarzschild metric we have been using is probably not the metric of most real black holes! This was known for a long time, but not until 1963 was the metric for a rotating black hole found by Kerr. For a black hole rotating in the ϕ direction the metric in Boyer-Lindquist coordinates is:

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - \frac{4aGMr \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 + a^2 \Delta \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2,$$

where

$$a = J/M$$

is the angular momentum per unit mass of the spinning hole,

$$\Sigma = r^2 + a^2 \cos^2 \theta,$$

and

$$\Delta = r^2 - 2GMr + a^2.$$

Actually the charged rotating black hole metric is nearly the same. Just replace Δ by

$$\Delta = r^2 - 2GMr + a^2 + Q^2/G,$$

where Q is the charge of the hole (in the proper units (esu)), and one has to add the c 's back in to use this equation.

Let's analyze this metric some. First note what happens if we take $a = 0$. Then $\Sigma = r^2$, and $\Delta = r^2 - rr_s$, so the metric becomes

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

which is just the Schwarzschild metric.

Next, notice the cross term, $dt d\phi$. In the tensor formulation this would correspond to a non-diagonal term in $g_{\mu\nu}$. It means that there is a blending of the the coordinates ϕ and t ; the forced motion we all must take in time will somehow be blended with motion in the ϕ direction. We will come back to this in a minute. Finally, note one needs caution using these coordinates since for $a \neq 0$ surfaces of constant t and constant r do not have the metric of a 2-sphere!

14.2 Horizon of Kerr metric: maximum rotation and charge of black holes

Next, how do we find the horizon, that is, the trapped surface or distance from which light can't escape? The proper way is to find the regions of the metric where all future lightcones converge. This requires more differential geometry than we have, but earlier for the Schwarzschild metric we found the horizon by when the sign of the dr term changed. Looking at the metric above, this will happen when the sign of Δ changes. Thus the horizon is at $\Delta = 0$. Solving $\Delta = r^2 - 2GM r + a^2 = 0$, we find two solutions for the horizon of a Kerr hole:

$$r_{\pm} = GM \pm \sqrt{(GM)^2 - a^2} = \frac{r_S}{2} \pm \sqrt{\left(\frac{r_S}{2}\right)^2 - a^2}.$$

These are called the inner and outer horizons, and the larger one r_+ , turns out to be the trapped surface. The smaller one is called the **inner or Cauchy horizon**. Note that if we take $a = 0$, $r_+ = r_S$ as expected, and that for $a \neq 0$, r_+ is always smaller than the Schwarzschild radius. Thus rotating black holes are smaller! Since there is no θ dependence in this equation we see that the horizons are both spherical.

Also note that if $a^2 > (GM)^2$, then there is no solution. A more careful analysis shows that when $a^2 > (GM)^2$ a black hole (trapped surface) cannot form. Thus there is a maximum angular momentum that a black hole can have:

$$a_{max} = GM = \frac{r_S}{2}, \text{ or } J_{max} = \frac{GM^2}{c}$$

This is called the **maximal black hole**. Also note that from the brief description of the charged rotating black hole above, there is also a maximal charge, which can be found from the condition that a horizon exists.

$$\left(\frac{GM}{c^2}\right)^2 \geq G \left(\frac{Q}{c^2}\right)^2 + \left(\frac{J}{Mc}\right)^2,$$

where I put the c 's back in to help with the units, each term is in units of meter², and cgs units must be used (with charge in esu). If this inequality is violated no horizon can form. The maximally charged black hole is therefore non-rotating, and the maximally rotating black hole is non-charged. In practice most of the Universe is not heavily charged, and for real black holes we expect the Kerr metric to obtain, and the maximally rotating black hole to be given by J_{max} above. Note for maximal black holes the horizon is a sphere of radius $r_+ = GM = r_S/2$, half the size of a non-rotating black hole of the same mass.

14.3 Singularity of Kerr metric

We would like to find the singularity in this metric, that is, the point that corresponds to $r = 0$ in the Schwarzschild metric. The proper way to do this is to calculate the curvature, as I discussed in a previous lecture, and then look for the places where it goes to infinity. The answer, which makes sense

from inspection of the metric is that the singularity occurs at $\Sigma = 0$, or $r_{\text{sing}}^2 + a^2 \cos^2 \theta = 0$. If $a = 0$, we have $r_{\text{sing}} = 0$ as we should, but if $a \neq 0$, we have the singularity only at $\theta = \pi/2$ and $r = 0$. This is peculiar, since why should it matter what θ is when $r = 0$? Actually there is a problem with the coordinates inside the black hole, which means that this is not really the right answer. Consider the region $\Sigma = 0$ in limit $M \rightarrow 0$, and $a \neq 0$. Then since there is no mass we know the metric must be equivalent to the Minkowski flat space metric; which it is in fact in spheroidal coordinates. Then $\Sigma = 0$ is just a coordinate singularity of the type we have seen before. So it takes some coordinate transformations and fairly subtle analysis to show that the actual singularity at $\Sigma = 0$ is actually a ring of radius a in the x-y plane. For maximal black holes both the horizon and $a = r_S/2$. Thus the ring singularity is just inside the horizon. If one entered the black hole at the equator one would hit the singularity immediately, but one entered from the north pole, one could all the way to the center without hitting the singularity.

14.4 The Ergosphere

Next, let's find a very interesting new property of spinning black holes. Consider a photon traveling in ϕ direction, so $ds^2 = 0$, and $d\theta = dr = 0$. The Kerr metric gives us the equation:

$$0 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - \frac{4aGMr \sin^2 \theta}{\Sigma} dt d\phi + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2.$$

Divide this equation through by dt^2 , and we get a quadratic equation for $\dot{\phi} = d\phi/dt$:

$$A\dot{\phi}^2 + B\dot{\phi} + C = 0,$$

where

$$A = \frac{[(r^2 - a^2)^2 - a^2 \Delta \sin^2 \theta]}{\Sigma} \sin^2 \theta,$$

$$B = -\frac{4GMa r \sin^2 \theta}{\Sigma},$$

and

$$C = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}.$$

This has solution:

$$\frac{d\phi}{dt} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Notice now that if $C = 0$, we have $\dot{\phi} = (-B \pm B)/(2A)$, which has $\dot{\phi} = 0$ as a solution. In this case we have that light itself, traveling in the ϕ direction stands still! (The other solution, $\dot{\phi} = -B/A$ is for light traveling in the same direction as the hole is spinning, while the solution $\dot{\phi} = 0$ is for light traveling opposite to the spin. If even light cannot move against the direction of the hole spin, then massive particles that must move slower than light obviously can't stay still no matter what they do. This "stationary light" solution occurs when $C = 0$, or $\Delta = a^2 \sin^2 \theta$, or $r_0^2 - 2GMR + a^2 - a^2 \sin^2 \theta = 0$, or $r_0^2 - 2GMR + a^2 \cos^2 \theta = 0$, or

$$r_0 = GM + \sqrt{G^2 M^2 - a^2 \cos^2 \theta}.$$

This is an ellipsoidally shaped region region that has a minimum radius of $r_0 = GM + \sqrt{G^2 M^2 - a^2}$ at $\theta = 0$ (north and south poles), and a maximum radius of $r_0 = 2GM = r_S$ at $\theta = \pi/2$ (equator). The

region inside this ellipsoid is called the **ergosphere**. Inside this region nothing can stand still. Everything rotates with the hole. The spinning hole actually drags the spacetime around it with it! Light itself going against the rotation direction will be carried backward around the hole. This is called “frame dragging”. If you drop something straight down into a spinning black hole, it will start orbiting the hole even though there is nothing but empty space outside the hole. Actually, this frame dragging occurs even outside the ergosphere, but it is possible to overcome it by using rockets. It actually occurs not just around black holes, but around any spinning object, since the Kerr metric is the description of spacetime in the space around any spinning object. It falls off as r^{-3} . Thus if you put a very precise gyroscope in orbit around the Earth, the frame dragging will cause it to precess. This is called the Lense-Thirring effect. The size of the precession is very small, only about 0.042 arcsec per year, but there was a NASA satellite called Gravity Probe-B, that was in orbit trying to measure this effect. It seemed to have failed due to experimental errors.

We can now picture a spinning hole from the top and from the side.

Fig: Spinning Black hole: Kerr metric: Ergosphere, and horizon

Considering a maximally spinning hole with $a = GM = r_S/2$, from the top there is the horizon at $r = GM = r_S/2$, just half the normal Schwarzschild radius. Then there is the ergosphere which from the top is a circle of radius $2GM = r_S$, but from the side goes from just touching the horizon at the north and south poles, to being out at r_S at the equator. We will discuss in a minute, why the ergosphere is called by that name.

14.5 Light orbits, inmost stable orbits for Kerr metric

In the same way we used the Euler-Langrange formalism to find the geodesics (orbit equations) for the Schwarzschild metric, we can find the geodesics for the Kerr metric. We can then go through the effective potential treatment and find the types of orbits. We won't spend the time to do this, but the result is that there are two closest stable orbits for massive particles and also two for light rays. Recall for the Schwarzschild metric the light orbit was at $r = 1.5r_S = 3GM$, and the closest stable orbit was at $r = 3r_S = 6GM$.

For the Kerr hole these orbits depend upon the value of angular momentum $a = J/M$, but for maximally rotating holes ($a = GM$) the light orbit for light rays going in the same direction as the spin is at $r = GM = r_S/2$ (**prograde light orbit**), This is the same r as the horizon and the singularity! For light rays going opposite to the spin the light orbit is at $r = 4GM = 2r_S$ (**retrograde light orbit**).

For massive particles around a maximally spinning hole, the innermost stable orbit is at $R = GM = r_S/2$ (prograde), and at $r = 9GM = 4.5r_S$ for retrograde orbits. Again the prograde (innermost stable circular orbit (ISCO) is at the same r as the horizon.

For reference, the effective potential for the Kerr metric is

$$V_{eff} = -\frac{\kappa GM}{r} + \frac{L^2}{2r^2} + \frac{1}{2}(\kappa - E^2)\left(1 + \frac{a^2}{r^2}\right) - \frac{GM}{r^3}(L - aE)^2,$$

where L is the conserved angular momentum per unit mass of the test particle (not J the angular momentum of the hole!), and E is the conserved energy per unit mass of the test particle, and where $\kappa = 0$ for light ray geodesics and $\kappa = 1$ for massive particles.

Recall that we calculated earlier the amount of energy that could be extracted by throwing garbage into a Schwarzschild black hole. We used the geodesic equations to find that a fraction of the rest mass could be converted to useful energy by gradually bringing the garbage down to the innermost stable orbit

and collecting the energy that had to be radiated away to get there. Finally the garbage goes down the hole, but we get out the difference between the energy $E = mc^2$ far away and the energy at that last stable circular orbit. For the Schwarzschild case, we found a rest energy fraction $f_{SC} = 1 - \sqrt{8/9} = 5.7\%$ could be obtained. Doing a similar calculation for a maximally spinning rotating black hole we find the maximal energy that can be extracted is a rest energy fraction of $f = 1 - \sqrt{1/3} = 42.2\%$ for the prograde inmost stable orbit and $f = 1 - \sqrt{25/27} = 3.7\%$ for the inmost retrograde stable orbit. Thus we have discovered an even more efficient way of producing energy. Throw garbage into a spinning black hole making sure to do it in the same direction as the hole is spinning. This is the most efficient energy producing device I have heard of.

14.6 Energy extraction from rotating black holes; ergosphere and Penrose process

A very interesting property of rotating black holes is that there exist negative energy particle trajectories. If one finds the geodesics and effective potential as described above one finds the energy to be: (Shapiro and Teukolsky equation 12.7.26)

$$E = \frac{2aGML + (L^2r^2\Delta + m^2r\Delta + r^3\dot{r}^2)^{1/2}}{r^3 + a^2r + 2GMa^2},$$

where E and L are the conserved quantities found from the Euler-Lagrange treatment of the Kerr metric. If one solves for the regions where $E < 0$, one finds that it corresponds to what we called the ergosphere. This property is why it is called the ergosphere, since *ergo* is the Greek word for energy. The ergosphere is also called the static limit, since particles cannot stay at rest inside this surface. The particles which are retrograde inside the ergosphere can have negative energy, but only particles inside the ergosphere can be scattered into geodesics with negative energies. Such negative energy particles always have trajectories that carry them into the black hole, so we can never see them. However, this gives rise to an interesting possible process called the Penrose process that allows one, in principle, to extract energy from a rotating black hole. Suppose, one shoots a massive particle inside the ergosphere, and then at a preprogrammed time it splits into two particles, one of which has negative energy and one of which has positive energy. Penrose showed that the negative energy particle would go down the hole, but the positive energy particle could escape, carrying with it more energy that it came in with!

$$E_{in} = E_{out} + E_{down},$$

or $E_{out} = E_{in} - E_{down}$, which implies $E_{out} > E_{in}$ since $E_{down} < 0$. This does not violate energy conservation, since it turns out the negative energy particle will slow down the spinning of the hole and reduce its energy. Thus this is a way to extract energy from a spinning hole. In fact, this particular process is probably not too useful astrophysically, because in order to get it to work the relative velocity of the pieces of the splitting particle must be greater than $c/2$.

Fig: Penrose process of extracting energy from rotating black hole

14.7 Inside the Kerr black hole

Recall that the singularity at $\Sigma = 0$ was a ring of radius a in the equatorial plane. In order to properly investigate the properties inside the Kerr black hole we need some coordinates analogous to the Kruskal-Szekeres coordinates for the Schwarzschild metric. These were found by Kerr and Schild in 1965. They

proved that singularity is a ring as described above and also showed that there were other regions inside just like in the Schwarzschild case.

They found that the ring singularity has some very weird properties. As one passes through it one goes through $r = 0$ to negative values of r ! And since r becomes timelike near the ring, for small values of r and θ near $\pi/2$, they found orbits in the ϕ direction that were timelike. Since ϕ a periodic function, this means looping around in ϕ near the ring means looping around in time; time becomes periodic inside the ring! That means that closed time-like loops must exist near the singularity. This is thus a time machine!

Even weirder would be the case the case discussed above where $Q^2/G + a^2 > G^2M^2$, and there would be no solution for a horizon. This would of course violate the cosmic censorship “no naked singularity” conjecture, but this is only a conjecture at this point. So in the pure Kerr solution, there would be no horizon, but there might still be a singularity! In this case, this would seem to be a “naked singularity”; that is a singularity not protected by a horizon. Because of the closed timelike loops, one can in principle make use of the causality violation occurring near the ring singularity to go “backwards in time” by an arbitrary amount as measured by t coordinate. However, people think that in real stars this can’t happen for reasons we will discuss below.

The full geometry of the Kerr black hole is more complicated and rich than the Schwarzschild geometry, so we want to simply the drawing of the Kruskal-Szekeres coordinates, by using what are called **Penrose diagrams**. Here we stretch time and space some and draw infinity as an edge. Lightcones stay the same, but the edges represent going all the out to infinity in time or space.

Fig: Penrose diagram of Schwarzschild Black hole

Recall region I is where we start and region II is inside the black hole; once here you have to hit the squiggly line which represents the singularity. Region III is the white hole, and region IV is another asymptotically flat region of space time, which you can’t get to without traveling faster than c .

Fig: Penrose diagram of Kerr Black hole

The Penrose diagram of the Kerr geometry starts out the same as the Schwarzschild diagram, but the inner horizon and the ring singularity makes a difference. The inside of the inner horizon ($r < r_-$) is split into two regions V and VI. Once you pass through the inner horizon you have a choice. You can either hit the singularity and die, or go through the ring and come out into other regions labeled V’ and VI’. Regions V and VI are still inside the black hole, but regions V’ or VI’ are other asymptotically flat spacetimes, similar to region IV. In these new asymptotically flat regions, the ring singularity is completely “naked” and has negative mass. With respect to the original region II, the singularity of course is inside the black hole, but for someone in region V’ or VI’ there is no black hole and it is possible to escape to any distance. At this point one can actually go through the ring again and get a region VII identical in structure to region III, and continue the process ad infinitum! In theory one can extend the structure downward as well through what used to be the white hole.

Note that in the Schwarzschild case one could not travel from region I to region III because the future lightcones all contained the singularity. In the Kerr metric however, this is not true! By going through the ring singularity one can reach another asymptotically flat spacetime without hitting any singularity. So is this a real wormhole? Can a rotating black hole be used as both a wormhole and a time machine? The answer is not yet proved one way or the other! The conjecture is that all the inner structure we have been discussing is unstable. An actual collapse of the spinning star is not perfectly symmetric or smooth. The “no hair” theorems say that one gets to the pure Kerr geometry only after the spacetime settles down. In particular, it is conjectured that the ring singularity is unstable and it is the inner horizon at $r = r_-$ that becomes the actual singularity. In this case, anything going into a Kerr black hole would

have to be destroyed. There are several reasons for this and Wald's book has a long description on page 318. One reason is that any small perturbations (light rays or particles) entering the black hole would get an infinite redshift by the time they get to the inner horizon. These would then have to be put into the stress-energy tensor and would destroy the perfect symmetry needed to maintain the ring singularity. The same thing would be true of anyone trying to go through the singularity into another spacetime. Their mass would change the metric so much as to destroy it.