Physics 161: Black Holes: Lecture 5: 22 Jan 2013

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5 Equivalence Principle, Gravitational Redshift and Geodesics of the Schwarzschild Metric

5.1 Gravitational Redshift from the Schwarzschild metric

We already discussed the curvature of time in the Schwarzschild metric. Clocks run slower in a gravitation field (at smaller values of r). This is one of the more measurable effects of GR. Measurements are possible because atomic transitions act like small clocks which send messengers (photons) to us here at Earth. Recall the proper time in the Schwarzschild metric:

$$d\tau = dt\sqrt{1 - \frac{2GM}{rc^2}},$$

where $d\tau$ is tick of a clock measured at r, and dt is the length of that same tick measured far away ($r \to \infty$).

Now consider some atomic transition, for example, the $n = 3$ to $n = 2$ transition of atomic Hydrogen. This is called the H-alpha transition and results in emission of a photon with energy $E_\gamma = 13.6\text{eV}(\frac{1}{3^2} - \frac{1}{2^2}) = 1.89$ eV. Using $\lambda = c/\nu$, and $E_\gamma = h\nu$, we find the corresponding wavelength $H\alpha : 6563.5$ **Angstroms**. Now in the frame at r, the photon has a fixed frequency ν_0, and each oscillation is like a little clock ticking with period $d\tau_0 = 1/\nu_0$. Thus the time between crests in the outgoing photon measures $d\tau$. That same photon travels out to us at $r = \infty$. Then that same period is observed as dt. Thus the $d\tau_\infty \equiv d\tau_{obs} = dt = d\tau_0/\sqrt{1 - 2GM/rc^2}$ is longer. Using the definitions above of ν and λ we see that the frequency decreases by the same factor and the more easily observed wavelength is longer at when observed at infinity than the emitted wavelength at r by

$$\lambda_{obs} = \frac{\lambda_0}{\sqrt{1 - \frac{2GM}{rc^2}}} = \frac{\lambda_0}{\sqrt{1 - \frac{r_S}{r}}},$$

where we used the definition of the Schwarzschild radius r_S.

The redshift of an emission line is defined as

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} = (1 - \frac{r_S}{r})^{-1/2} - 1,$$

where λ_0 is the emitted wavelength in the rest frame of the atom. Note for small values of $2GM/rc^2 = r_S/r$, this expression can be Taylor expanded as $z \approx GM/rc^2 = \frac{1}{2}r_S/r$.

This gravitational redshift is totally different from either the normal Doppler shift redshift caused by an atom moving away from the observer, or the cosmological redshift caused by the expanding Universe. Let's see how big the effect is by calculating it for some typical astronomical objects. First how about emission from the surface of the Sun? We expect the wavelengths recieved by us on Earth to be longer than the emitted wavelengths since the emitted photons have to climb out of the potential well of the Sun where clocks run slower. Using the approximation above $z_\odot \approx \frac{1}{2}3\text{km}/7 \times 10^5\text{km} \approx 2 \times 10^{-6}$. This is not a very big effect; a clock on the surface of the Sun loses about 30 seconds/year. For H-alpha the line redshift would be only 0.014 Angstroms.

How about for a white dwarf star? The mass of a white dwarf is about $0.6M_\odot$, and the radius is about 5500 km, near to the Earth radius. Thus $z \approx \frac{1}{2}(.6)(3\text{km})/5500\text{km} \approx 1.6 \times 10^{-4}$, and the wavelength of H-alpha emission recieved on Earth would be $6563.5(1+z) = 6564.6$ Angstroms, quite easily distinguishable from local H-alpha lines. This result agrees well with the measurements from the nearby white dwarf star Sirius B. A clock on the surface of Sirius B would lose about 0.6 second/hour.

Next consider the case of a neutron star, with $M = 1.4M_\odot$ and $R = 10$ km. Now $r_S = (1.4)(3\text{km}) = 4.2$ km, so really should not use the approximation, but calculate $z = [1-(4.2\text{km}/10\text{km})]^{-1/2} - 1 = 0.313$. This is big shift: the H-alpha line would appear at 8618.3 Angstroms and clocks would run 31% slower.

Finally what about light emitted from the Schwarzschild radius of a black hole? With $r = r_S$, $z \to \infty$! Light would be invisible since it would be redshifted to infinitely low energies. Clocks sitting at that location would appear to have stopped!

5.2 Light bending and the Equivalence Principle

To find proper answers in GR we calculate using the metric, in our case the Schwarzschild metric, but there is an interesting, more intuitive way that uses the Equivalence Principle directly. Let's look briefly at two examples. Imagine someone in a falling elevator. The equivalence principle says that experiments done in this falling frame will give the same results as experiments done floating in free space far from any massive object. Suppose the guy in the falling elevator shines a laser beam horizonally from one side of the elevator to the other. That light must follow geodesics and therefore go "straight" across the elevator; if the beam leaves from a height of 3 feet above the elevator floor, it must arrive on the other side also three feet above the floor. From the point of view of the elevator man, the light must travel exactly horizontally. But now think of someone watching through the (glass) walls of the elevator. The man shoots the laser from a height of 3 feet above the floor, but then while the light is traveling across, the elevator continues to fall. Thus when the light hits the opposite wall 3 feet above the floor, the wall has moved down. Therefore the light cannot go straight in the Earth frame. The light must also fall. This gedanken experiment shows the light must fall in a gravitational field in the same way as everything else. Of course, from the point of view of GR, light is just following the geodesic through a curved spacetime; we will calculate those soon, but here we forsee what that calculation will give. In particular, light from distant stars travelling near the limb of the Sun must be "deflected" towards the Sun. Sir Author Eddington did this experiment in 1919, and it garnered world wide fame for Albert Einstein, because it was one of the first proofs of Einstein's GR theory. In the homework, I am asking you to calculate how far the light falls in a gravitational field, that is how much it bends.

5.3 Gravitational Redshift again

The equivalence principle can also be used to show that light must gravitationally redshift. Again consider our man in a falling elevator. This time he shoots his laser from the floor to the ceiling.

First think about it from the point of view of someone outside watching through the glass walls. In the Earth frame the freely falling elevator picks up speed while the light is in transit, and so there is Doppler shift caused by the speeding elevator ceiling. The time the photons spend in transit is $t = h/c$, where h is the height of the elevator, and the speed increase of the elevator is $\Delta v = gt = gh/c$, where $g = GM_{\text{Earth}}/r^2 = 9.8\text{m/s}^2$. We could find the expected Doppler shift from the relativistically correct formula $\nu_{obs} = \nu_0 \sqrt{(1 + \Delta v)/(1 - \Delta v)}$, but a Taylor expansion of this is good enough for the slow speeds involved here: $\nu_{obs} \approx \nu_0(1 + \frac{1}{2}\Delta v)(1 + \frac{1}{2}\Delta v) \approx \nu_0(1 + \Delta v)$, or $\Delta\nu = \nu_{obs} - \nu_0 = \Delta v \nu_0$ Thus in the Earth frame we expect the light beam to be blue shifted by $\Delta\nu/\nu_0 = \Delta v/c = gh/c^2$, where I put the $c's$ back in. However, the equivalence principle says that this cannot be! The man in the elevator can't tell whether he is falling or floating freely in deep space, so he expects correctly that the frequency of light recieved at the elevator's ceiling is the same as the frequency he sent from the floor. Thus there must be another effect to cancel this one! The light traveling up must be gravitationally redshifted to counterbalance this effect. Thus there must be a factor of $\Delta\nu/\nu_0 = -v/c = -gh/c^2 = -GMh/r^2c^2$. This is not quite the result we got before from the metric, since there is a factor of r^{-2} rather than r^{-1}, but before we found the redshift between $r = r_0$ and far away ($r = \infty$) and here we found it only between r_0 and r. So we can continue with this derivation by integrating the small redshift we found $\Delta z \sim \Delta\nu/\nu_0$ from r_0 to ∞.

$$ z \sim \int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \approx \int_{r_0}^{\infty} -\frac{GM}{r^2c^2}dr. $$

This gives $\ln(\nu_\infty/\nu_0) \approx -GM/r_0c^2$, or $\lambda_\infty = \lambda_0 \exp{(GM/r_0c^2)}$. This is still not the correct answer, but if we assume the factor in the exponential is small and Taylor expand the exponential using $e^x \approx 1 + x + x^2/2 + ...$, we do find $\lambda_\infty/\lambda_0 \approx 1 + GM/rc^2$, or $z = GM/rc^2 = \frac{1}{2}r_S/r$, the Schwarzschild result for small z.

5.4 Geodesics of Schwarzschild metric from Euler-Langrange

Let's return to our work on geodesics and apply the Euler-Lagrange equations to find the geodesics of the Schwarzschild metric. Remember that the Schwarzschild metric is the unique metric around stationary, spherically symmetric, uncharged objects, so what these geodesics do is tell us how thing move around the Earth, around the Sun, and around uncharged, non-spinning black holes. These are the General Relativistic extension of Newton's and Kepler's laws of motion. Recall the Schwarzschild metric is

$$ ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. $$

As above, we set $c = 1$, and take affine paramter $\lambda = s = \tau$, and extremize $s = \int L d\tau$, with

$$ L = 1 = \left[\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{1/2} $$

The Euler-Lagrange equation for t is then

$$ \frac{d}{d\tau}\frac{\partial L}{\partial\dot{t}} - \frac{\partial L}{\partial t} = 0, $$

which since $\partial L/\partial t = 0$, implies there is a conserved quantity we will call energy per unit mass:

$$ \frac{\partial L}{\partial\dot{t}} \equiv \frac{E}{m}. $$

Calculating,

$$\frac{\partial L}{\partial \dot{t}} = \frac{1}{2}[\cdots]^{-1/2}(1 - \frac{2GM}{r})2\dot{t},$$

where we abbreviated $L = [\cdots]^{1/2}$. Using $L = 1$, we find our t equation

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = \frac{E}{m}.$$

We don't yet know that the constant has anything to with energy, but we call it E/m, because of our experience with the Minkowski metric. For $r \to \infty$, the Schwarzschild metric goes to the Minkowski metric and for the Minkowski metric $\dot{t} = E/m$.

Next we find the ϕ equation:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0,$$

so again we have a conserved quantity $p_\phi = \partial L/\partial \dot{\phi} \equiv -l/m$, where we will call this conserved quantity the angular momentum per unit mass. Recall Noether's theorem which says if physics is unchanged by a rotation then angular momentum is conserved. Since the metric does not depend explicitly on the angle ϕ, we get that result here. Doing the differentiation we find

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}[\cdots]^{-1/2}(-r^2 \sin^2 \theta 2\dot{\phi}) = -r^2 \sin^2 \theta \dot{\phi}.$$

Thus our ϕ equation reads

$$\frac{l}{m} = r^2 \sin^2 \theta \dot{\phi}.$$

Note that it makes sense that we called the constant of motion l/m, since this matches the normal definition of angular momentum, $l = \vec{r} \times \vec{P}$, with $v = r \sin \theta \dot{\phi}$.

Next, we consider the θ equation. Here we find that

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \neq 0,$$

thus we do not have a conserved quantity for this equation. We find

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}[\cdots]^{-1/2}(-r^2 \dot{\phi}^2 2 \sin \theta \cos \theta) = -r^2 \dot{\phi}^2 \sin \theta \cos \theta,$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}[\cdots]^{-1/2}(-r^2 2\dot{\theta}) = -r^2 \dot{\theta}.$$

Thus our θ equation reads:

$$\frac{d}{d\tau}(r^2 \dot{\theta}) = r^2 \dot{\phi}^2 \sin \theta \cos \theta.$$

Finally we come do the r equation, which is kind of messy because of all the explicit r dependence in the metric. However, we don't have to do it, because we can get the fourth equation we need to specify the equations of motion from our definition of the Lagrangian, $L^2 = 1$:

$$1 = (1 - \frac{2GM}{r})\dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2GM}{r})} - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2.$$

4

Since our object and metric are spherically symmetric we can simplify things by only considering motion in the equatorial plane ($\theta = \pi/2$, and $\dot{\theta} = 0$). Of course we have to remember that we made this assumption later when we use our equations! If we try to consider motion that has a changing θ, or which is not in this plane we would need to come back to the equation above. In this case then, the equation $L = 1$ becomes:

$$1 = (1 - \frac{2GM}{r})\dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2GM}{r})} - r^2\dot{\phi}^2.$$

Now eliminate variables other than r, using the constants of motion we have from the above equations: $l/m = r^2\dot{\phi}$, and $E/m = (1 - \frac{2GM}{r})\dot{t}$, to get:

$$1 = \frac{E^2}{m^2(1 - \frac{2GM}{r})} - \frac{\dot{r}^2}{(1 - \frac{2GM}{r})} - \frac{l^2}{m^2 r^2}.$$

We can write this in a nicer form by solving for $m\dot{r}^2$,

$$m\dot{r}^2 = \frac{E^2}{m} - \left(m + \frac{l^2}{mr^2}\right)(1 - \frac{2GM}{r}),$$

Remembering the definition of $\dot{r}$ and using dimensional analysis to put back the c's, we can write this as:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)\left(mc^2 + \frac{l^2}{mr^2}\right).$$

Notice that, as I mentioned in my handwaving description, a step in proper time $d\tau$ forces a step dr in the r direction. Thus this geodesic equation shows that things fall due to the spacetime curvature of the metric. We will look at these geodesic equations in some detail, but for now let's just take one limit of this last equation.

Suppose the angular momentum $l = 0$, which we might expect for radial infall towards a spherical mass. Our equation then is:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)mc^2 = 0.$$

Next consider a case where you start at rest far from the object so that at proper time $\tau = 0$, $m(dr/d\tau)^2 = 0$, and $r \to \infty$. Our equation becomes:

$$\frac{E^2}{mc^2} - (1 - \frac{2GM}{\infty})mc^2 = 0,$$

or $E^2/mc^2 = mc^2$, or simply $E = mc^2$! So at $\tau = 0$ the total energy is just $E = mc^2$. In Newtonian mechanics the energy at infinity is usually defined as $E = 0$. Isn't it nice how these important results are just built right into the general relativistic view of spacetime. Also since energy E is conserved along geodesics we know that $E = m$ always. This E is not the Newtonian energy; it is *the* conserved quantity, which is better than the sum of $\frac{1}{2}mv^2 + V$. Finally note that if we would have started with some velocity at $r = \infty$ then $E > mc^2$ but it still would have been conserved.

At later times during this radial infall from rest, our equation becomes:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - mc^2 + \frac{2GM}{rc^2}mc^2 = \frac{2GMm}{r}.$$

That is simply

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0,$$

which looks remarkably like the Newtonian case $\frac{1}{2}mv^2 - GMm/r = 0$! But this is not the same equation. It is fully relativistic and the time derivative is τ not t (remember $t = \gamma\tau$ in SR). In general you integrate these 4 equations to get the complete picture of motion near the Earth, Sun, or Black Hole. We will come back to these soon.