

Physics 161: Black Holes: Lecture 8: 31 Jan 2013

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8 Extracting Energy from a Black Hole; Light orbits

8.1 Extracting Energy from a Black Hole

Now let me tell you why black holes are the most efficient energy generation devices we have yet conceived. Much better than nuclear fission or even nuclear fusion. Of course, there is the small problem of not having a black hole near by and available for use.

First consider throwing some junk into a black hole from far away. We learned before that the total energy of the stuff far from the black hole is just $E = mc^2$, where m is the mass of the stuff. Suppose this stuff doesn't go straight into the hole, but is slowed down in a series of orbits, finally settling into the last stable orbit we just discussed. What is the energy of the stuff now? We can use the radial geodesic equation to find out. On a circular orbit $dr/d\tau = 0$, so we have

$$\frac{E^2}{m} = \left(1 - \frac{r_S}{r}\right) \left(m + \frac{l^2}{mr^2}\right).$$

Next we plug in $l^2 = 3r_S^2 m^2$, and $r = 3r_S$,

$$\frac{E^2}{m} = \left(1 - \frac{1}{3}\right) \left(m + \frac{3r_S^2 m^2}{m9r_S^2}\right) = \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{3}\right) m = \frac{8}{9} m.$$

Thus

$$E = \sqrt{\frac{8}{9}} mc^2.$$

Somehow during the transition from far away to the last stable orbit the energy has decreased. Note that in doing this we did *not* follow a geodesic. E is conserved along geodesics, so starting at $r = \infty$ means $E = mc^2$ still at $r = 3r_S$. Thus the stuff must accelerate to get to the smaller stable circular orbit. Typically falling stuff hits other falling stuff and the collisions are what cause the acceleration. Thus, the original energy must have radiated away, and therefore be available for other use. The fractional amount of energy radiated away is $(mc^2 - E)/mc^2 = 1 - \sqrt{\frac{8}{9}} = 5.7\%$.

Let's compare this with other forms of energy generation. When burned, a gallon of gasoline produces 1.32×10^8 Joules of energy and has a mass of 2.8 kg. Thus the fractional mass energy lost when a gallon of gasoline is burned is $\Delta E/E = (1.32 \times 10^8)/((2.8)(3 \times 10^8)^2) = 5.2 \times 10^{-10}$. This is thus also the fraction of the rest mass of the gasoline that is turned into useful energy, a far cry from the 5.7% we get from the black hole.

Next, consider nuclear energy, say the complete fission of 1 kg of pure Uranium 235. This produces 8.28×10^{13} Joules, so $\Delta E/E = (8.28 \times 10^{13})/((1)(3 \times 10^8)^2) = 0.0092\%$, still below the black hole. Finally one usually thinks of the main source of energy in the Universe as nuclear fusion of Hydrogen into Helium (as in the Sun). Actually in large stars, the Helium is then burned to Carbon, Oxygen, etc. all the way up to Iron, which is the most stable element. This whole chain of nuclear fusion releases about 0.9% of the rest mass into usable energy, still more than a factor of six below what you get from throwing your garbage into a black hole. Black holes are the best devices we know of for producing energy. In fact, using a rotating black hole we can turn even more of the rest mass of stuff into useful energy. The energy generation method discussed here is thought to be the power source of quasars, the most luminous and energetic steady objects in the Universe.

8.2 Geodesics and motion of light around a black hole

In one sense light is just the same as a particle with its rest mass $m \rightarrow 0$. To find the geodesics for light, then, we should be able to just take the limit $m \rightarrow 0$ in the geodesic equations we already found. As a reminder I reproduce those equations below, in the special case with $\theta = \pi/2$ relevant for motion in the x-y plane.

$$m \left(\frac{dr}{d\tau} \right)^2 = \frac{E^2}{m} - \left(1 - \frac{r_S}{r} \right) \left(m + \frac{l^2}{mr^2} \right)$$

$$\frac{d\phi}{d\tau} = \frac{l}{mr^2}$$

$$\frac{d\theta}{d\tau} = 0$$

$$\frac{dt}{d\tau} = \frac{E}{m} \left(1 - \frac{r_S}{r} \right)^{-1}.$$

But we immediately see that if we try to take the limit $m \rightarrow 0$ in these equations we get into trouble. We get nonsense like $0 = \infty$, etc. Actually we talked about this trouble before when discussing light. For light the proper time $d\tau$ is always zero. Thus when we chose to use τ for our affine parameter, we excluded the use of these equations for light! What we should do is go back to the Schwarzschild metric, choose a different affine parameter, and use the Euler-Lagrange equations to rederive the geodesics for light. However, we can instead use a trick. This is another of those useful tricks worth learning.

We note that in the limit $m \rightarrow 0$, the speed of the particle goes to c and relativistic factor $\gamma \rightarrow \infty$. That is why the proper time $\tau \rightarrow 0$. However, the combination $m\gamma$ is the energy of the particle and stays fine as $m \rightarrow 0$. Thus if we use $\lambda = \tau/m$ as the affine parameter everything may work out fine. We do that by substituting $\tau = m\lambda$ everywhere in the geodesic equations above, and then multiplying through by m :

$$\left(\frac{dr}{d\lambda} \right)^2 = E^2 - \left(1 - \frac{r_S}{r} \right) \left(m^2 + \frac{l^2}{r^2} \right)$$

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}$$

$$\frac{d\theta}{d\lambda} = 0$$

$$\frac{dt}{d\lambda} = E \left(1 - \frac{r_S}{r} \right)^{-1}.$$

Now we can take the limit $m \rightarrow 0$ and get equations that make sense. Only the r equation changes to

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{r_S}{r}\right) \frac{l^2}{r^2}.$$

Note that the trick we did was to try to rescale our original equation that had a bad affine parameter τ to a different affine parameter that behaved ok. We scaled by a constant, such that we could take the limit we wanted. If the equations we got didn't make sense we might have tried another rescaling. The basic idea of rescaling differential equations is a trick that is often useful when you want to take some limit of the equations.

Note one problem with the above is that we can't use the old definitions of E and l which involve m . We know from the Euler-Lagrange treatment that these quantities are conserved along geodesics because there is no explicit t or ϕ dependence in the metric, so we can still call them energy and angular momentum, but we don't know yet what the formulas for them are when considering light geodesics.

OK, we are now ready to do the effective potential treatment for the photon geodesic equation. We count $(dr/d\lambda)^2$ as the kinetic energy term, and therefore find that

$$V_{eff} = \left(1 - \frac{r_S}{r}\right) \frac{l^2}{r^2} = \frac{l^2}{r^2} - \frac{r_S l^2}{r^3}.$$

Plotting this we can discover the light orbits around a black hole.

Fig: Effective potential for light orbits in Schwarzschild metric

Using what we learned from the massive case we see that there are three types of orbits: an unstable circular orbit, a coming in, then out orbit, and a capture orbit. The really new thing here is the circular light orbit. In Newtonian mechanics light cannot orbit anything, but here in GR it can! Let's find the radius of the light orbit:

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{r^3} + \frac{3r_S l^2}{r^4} = 0,$$

or

$$r_{lightorbit} = \frac{3}{2}r_S.$$

Thus at $1.5r_S$, just half the last stable orbit for massive particles, light will orbit the hole. Since it is an unstable orbit, a black hole can trap light for some amount of time, and then release it back out to infinity.

Light orbits can produce some pretty wierd effects. Suppose an advanced race built a tube around a black hole just at the light orbit radius. This is outside the trapped surface, so people could go in that tube and still get away from the black hole. If you were standing in such a tunnel, what would you see? The light would go round and round, and you would see the back of your head. Actually further down the tube you would see yourself again. The tube would in fact look infinitely long, and perfectly straight. If you tried to measure its length with a laser you would never find the end. So by looking you would not know that you were in a circular tube! It would look perfectly straight to you!

Even weirder would be if you built a such a tube a tiny bit inside the light orbit radius $r = 1.5r_S$. In this case if you shone your laser beam, it would curve inward towards the hole and hit the inner wall of the tube. Thus to you it would seem that the inner wall was further out than the center. The appearance would be that the entire tube was curving outward, not inward! However, if you got on your skate board and gave yourself a push, you it would not take any energy to coast around the hole, it is all at the same

r . It would look like you were going uphill, but feel like you were going level! Finally, what would it be like if you built the tube just outside the light orbit radius? Check out the effective potential; light trying to circle just outside the peak in the potential (light orbit radius) will “fall” out to infinity. Thus the light would go for a way down the tube but eventually hit the ceiling. Thus it would look like the tube case curving down towards the hole (which it is). However, again a skateboard ride would feel like you are going straight, even though it looks like you are going downhill.