Physics 161: Black Holes: Lecture 10: 7 Feb 20113

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10 Some Questions and Puzzles Around Black Holes

10.1 How fast is an object going when it enters a black hole?

Suppose we take the case where someone starts from rest at ∞ and falls into the hole. We have already calculated this case. Of course, as viewed by the person falling, there is no speed at all. But as viewed from far away the person's time slows down and then stops as it enters the Schwarzschild radius. The calculation is done starting from our radial and time equations: $dr/d\tau = \pm\sqrt{\frac{2GM}{r}} = \pm\sqrt{\frac{r_S}{r}}$, and $dt/d\tau = (E/m)(1-\frac{r_S}{r})^{-1} = (1-\frac{r_S}{r})^{-1}$, where we used conserved energy $E = m$ which is valid starting at rest from infinity. This $dt/d\tau$ equation might seem funny since we have often seen instead $dt/d\tau = (1 - \frac{r_S}{r})^{-1/2}$ coming from the metric proper time at a fixed value r, but the τ in these cases differ. One is for a stationary observer at fixed r, and the other is the proper time for someone falling into the hole from infinity. Dividing these equation we find the relation between r and t, that is the speed as seen from from away:

$$v_{\text{far away}} = \frac{dr}{dt} = -\left(1 - \frac{r_S}{r}\right)\sqrt{\frac{r_S}{r}}.$$

We see again that as $r \to r_S$, $v \to 0$. The far away observer never sees the person fall in. Following Wheeler, we define the **shell frame** as the frame that is static at a given value of r, that is someone who is just hovering over the hole at certain radius. We on Earth are in a shell frame, since the Earth's surface holds us here.

How does it look from someone on a shell near the horizon at $r = r_S$?

Distances are measured in this frame by poking a meter stick in the radial direction, and thus are just the proper distances dl_r from the Schwarzschild metric:

$$dl_r = dr_{\text{shell}} = dr \left(1 - \frac{r_S}{r}\right)^{-1/2},$$

where $r_S = 2GM/c^2$ as usual. How does time run in the shell frame? Well, r is not changing between clock ticks, so compared to far-away time t, the metric value of $d\tau$ is the shell frame time:

$$dt_{\text{shell}} = dt \left(1 - \frac{r_S}{r}\right)^{1/2}.$$

Using the two equations above, we can relate times and distances in the shell frame to those in the far away frame (e.g. Schwarzschild coordinates r and t). We can also find the speed measured in the shell

frame:

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = (1 - \frac{r_S}{r})^{-1}\frac{dr}{dt}.$$

For the person falling in from far away, we put in the result for dr/dt above to find:

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = -\sqrt{\frac{r_s}{r}}.$$

This gives the remarkable result that to a shell observer, sitting at $r = r_S$ (well, just above that so they can tell us what happened!), the falling objects goes by at $v_{\text{shell}} = 1$, that is the speed of light! Isn't it strange that the same object doing the same thing can be moving at 0 speed or c from different vantage points.

We have to ask now what is the energy of that falling object. From far away it is $E = m$ and this stays the same; energy is conserved along geodesics according to our Euler-Lagrange calculation. According to the shell observer, they see something going be at nearly the speed of light, that is $\gamma \to \infty$, so $E = m\gamma \to \infty$. This is actually consistent with what we said before that the redshift of something escaping from the Schwarzschild radius was infinite. That is, it would take an infinite amount of energy to get out, and the escape velocity is c. We once again that in special and general relativity, the velocity and energy of an object are frame dependent, and there is no such thing as the "real velocity" or "real energy" of an object. Also note that in General Relativity it is usually impossible to divide up the energy into kinetic and potential energy in a meaningful way. These are frame dependent also (because velocity is).

There is yet one other velocity to think about. The mixed coordinate speed $dr/d\tau$ which we used above. This is how fast the falling person sees the outside world go by: $dr/d\tau = \pm\sqrt{\frac{2GM}{r}} = \pm\sqrt{\frac{r_S}{r}}$. Wheeler calls this the **rain frame**. Here we see again the speed goes to c at the horizon. Also we see that inside the horizon the speed becomes faster than light! Is this really true? Well the formula is correct, but the interpretation of r changes inside. So there is not really motion faster than c.

10.2   What does it look like standing near or falling into a black hole?

Using our light bending formulas we can describe how things look to someone falling into a black hole, or to someone sitting on a shell at a distance r. We have to be careful what variable we use. So far we mostly discussed things from the point of view of the "far-away" or Schwarzschild coordinate frame. We also discussed the "free-fall" frame. What is the metric in the free fall frame? Minkowski of course; that's called the equivalence principle. There is one other frame we've kind of discussed which is useful. The frame we have here on Earth, sitting stationary at a fixed distance from the center.

So let's figure out what a black hole "looks" like when we are standing still above it at a fixed value of r, i.e. in the shell frame. For someone a coordinate distance r from the hole, looking toward the hole, what will they see? Let's define the angle of looking from directly overhead, that is $\theta = 0$ is looking straight away from the hole and $\theta = \pi$ is looking directly at its center. Looking straight at the center of the hole one sees only blackness, since no light can come out of it (assuming that there is no one else between you and the hole). At a certain angle one would see the "edge" of the black hole, that is a separation between the blackness of the hole and the sky around it. Now because light travels on curved geodesics, one cannot use normal Euclidian geometry to find that angle. One must use the light geodesic equation we derived last time. To find the angular size of the hole we want to find those geodesics that just barely make it into the hole, and those that just barely miss the hole. When we look out at a certain

angle, we are specifying a light geodesic. We want to map these angles into the various geodesics and see which go into the hole and which don't. We found earlier that the light geodesics are specified only by the "impact parameter" $b = l/E$, and r_S, so there should be some mapping between angle of looking and b.

We can tell geodesics which go into the hole and which don't using the effective potential for light, which we found from the radial geodesic equation. Remember that for given values of E and l (actually only the combination $b = l/E$ occurs) there are light paths that come in towards the black hole, reach a point of closest approach (which we call the turnaround radius r_{ta}), and then return to $r = \infty$. There are also geodesics that go over the hump and plunge into the black hole, and one geodesic at the light orbit radius, $r_{\text{lo}} = \frac{3}{2} r_S$ that circles the black hole.

It seems clear that if the turnaround radius is outside the light orbit radius, then the light will escape the hole, and thus the critical geodesic, the light path that just goes in and therefore defines the visible "edge" of the black hole, is the one that has $r_{\text{ta}} = r_{\text{lo}}$, or $r_{\text{ta}} = \frac{3}{2} r_S$.

We find this critical geodesic by first finding the turnaround radius from our radial geodesic equation,

$$\frac{dr}{d\lambda} = \left[E^2 - \left(1 - \frac{r_S}{r} \right) \frac{l^2}{r^2} \right]^{1/2},$$

by setting $dr/d\lambda = 0$. This gives $E^2 = (1 - r_S/r_{\text{ta}}) l^2 / r_{\text{ta}}^2$, or

$$b = \frac{r_{\text{ta}}}{\sqrt{1 - r_S/r_{\text{ta}}}},$$

where we used the definition $b = l/E$. We find the critical geodesic, which is specified by b_{crit}, the critical value of b, by setting the turnaround radius to its smallest possible value, the light orbit radius. Thus we find $b_{\text{crit}} = \frac{1}{2}\sqrt{27} r_S = 2.598 r_S$.

Now we need a way to map this value b_{crit} into a critical viewing angle θ_{crit}. First we want to map angles in coordinate space, r and ϕ, to angles in the shell frame. Consider someone sitting still on a shell at r. Consider a light ray sent out from event A at r to event B at $r + dr$ at some angle θ.

Fig: Angles of light in Schwarzschild and shell frames

What does this do in the shell frame? Well since the Schwarzschild metric is flat in the angular directions, know that that $dl_\phi = r d\phi$ will be the same to both the far away and shell observers. We also know from the metric and definition of proper length that $dr_{\text{shell}} \equiv dl_p = dr/\sqrt{1 - r_S/r}$ is longer than dr. Thus the angles will be different. Thus locally we can write $\tan\theta_{\text{shell}} = r d\phi / dr_{\text{shell}}$.

Next, we can use our light bending formula, the combination of radial and ϕ geodesics we found last time, to relate these quantites to b,

$$\frac{d\phi}{dr} = \frac{\pm 1}{r^2 \sqrt{\frac{1}{b^2} - \left(1 - \frac{r_S}{r}\right)\frac{1}{r^2}}}.$$

Combining the above two equations we find:

$$\tan\theta_{\text{shell}} = \frac{r d\phi \sqrt{1 - r_s/r}}{dr} = \pm \frac{\sqrt{1 - r_s/r}}{\sqrt{(r^2/b^2) + (r_s/r) - 1}}.$$

This is good for any value of r and b, but we can specialize to the critical value of b to find the critical angle

$$\tan\theta_{\text{crit}} = \pm \frac{\sqrt{1 - r_s/r}}{\sqrt{(4/27)(r^2/r_s^2) + r_s/r - 1}}.$$

We can now use these formulas to find the angular size of a black hole as we hover over it. First plug in the light orbit radius $r = \frac{3}{2}r_S$, to find $\tan\theta_{\text{crit}} = \infty$, or $\theta_{\text{crit}} = 90^0$. This is just as expected, since looking out at 90^0 means looking straight out over the horizon which is the light path. We know light with $\theta_{\text{shell}} = 90^0$ or more will never get out, but that light sent at slightly less angle will get out. Thus below you, like an infinite black plane, you see blackness (the hole). Looking out in any direction you see yourself at a distance of $3\pi r_S$, and then another ring of you's at twice this distance, and then again another ring, and so on forever! Above you the entire sky is displayed. Even stars behind the black hole can be seen, since curved light paths bring them to you. The entire celestial sphere is projected onto 1/2 the sky.

Going further out than $r = 1.5r_S$ we will see stars in more than one-half the sky, and the black hole in less. Light launched at angle from the vertical less than the θ_{shell}(critical) will escape the black hole, while other light rays will be captured. You just reverse these light rays to know what the shell observer sees.

For example, using our equation above, (or integrating the light bending equation shows) that at $r = 2.5r_S$, $\tan\theta_{\text{crit}} = \pm 1.356$. Now there are actually several angles that satisfy this value of $\tan\theta_{\text{crit}}$, since arctan is a multivalued function and there is the $\pm$ from when we took the square root. The easiest thing is to use common sense. Naively taking the inverse tangent gives $\theta = \pm 54^0$, neither of which makes sense, since at this radius we expect the black hole to be smaller than half the sky. But adding 180^0 to -54^0 is also a solution and gives $\theta_{\text{crit}} = 126^0$ which makes sense and is the correct answer. Plug in other values for yourself: for example at $r = 1.08r_S$ the angle is 40^0, while for $r = 1.001r_S$ the angle is 4.7^0. Thus as you move closer to the black hole than the light orbit radius, it seems to cover more and more of the sky, almost completely enveloping you as you approach the Schwarzschild radius. The entire sky is still visible, but mapped into a small cone directly above your head. Finally as you cross the horizon to inside the black hole that small opening closes up; at that point there is no shell observer possible and so our equations no longer apply.