→ Adiabatic Invariants and Action-Angle Variables

c) Adiabatic Invariants

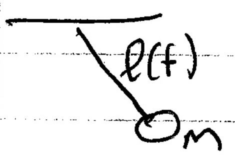
(Bounded phase space ⟹ QPO)

→ Consider finite motion in 1D. Motion characterized by λ parameter, such that:

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \frac{1}{\tau}$$

↳ period of motion

i.e.



$$\frac{1}{\ell}\frac{d\ell}{dt} < \sqrt{g/\ell}$$

pull on string

thus, $\dot{E}$ will be "small / slow" (i.e. $H = H(\lambda(t), p, q)$)

↑ parametric dependence,

Now, $$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$$

as λ varies slowly compared to $\omega_0 = 1/\tau$, can average over t on fast scales, i.e.

$$\frac{d\bar{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}} = \overline{\frac{\partial H}{\partial \lambda}}\frac{d\lambda}{dt}$$

↳ avg. over motion q, p → fast

where $\overline{A} = \frac{1}{T}\int_0^{\tau} dt\, A(t)$ → holding E, λ fixed!

⇒ $$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^{\tau} \frac{\partial H}{\partial \lambda}\, dt$$

Now, $$\dot{q} = \frac{\partial H}{\partial p}$$

$$dt = \int dq / \partial H/\partial p$$

∴ can take $\int dt \rightarrow \oint \frac{dq}{\partial H/\partial p}$

$\oint$ ↔ complete circuit orbit

so finally,

avg.
↓

$$\frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\left\{\frac{\oint (\partial H/\partial \lambda)\, dq / (\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}\right\}.$$

↑
τ

Now: – integrations must be performed for fixed, given value of λ (i.e. $\dot{\lambda}/\lambda \ll \omega_0$)

– on such path (n.b. why "path" of interest!), $H = E$ and $p = p(q; E, \lambda)$

$\therefore \quad H(p, q, \lambda) = E$

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0$$

$$\Rightarrow \quad \frac{\partial H/\partial \lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$$

$$\therefore \quad \frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\,\frac{\oint -(\partial p/\partial \lambda)\,dq}{\oint dq\,\partial p/\partial E}$$

$(1/\partial H/\partial p = \partial p/\partial E)$ (fixed λ)

so, re-writing:

$$\frac{d\bar{E}}{dt}\oint dq\,\partial p/\partial E + \frac{d\lambda}{dt}\oint(\partial p/\partial \lambda)\,dq = 0$$

$$\Rightarrow \oint_{\substack{E,\lambda \\ \text{fixed}}} dq \left\{ \frac{\partial p}{\partial E} \frac{dE}{dt} + \frac{\partial p}{\partial \lambda} \frac{d\lambda}{dt} \right\} = 0$$

$$\Rightarrow \boxed{\overline{dI/dt} = 0}$$

where $I = \oint_{\substack{E,\lambda \\ \text{fixed}}} \frac{p\,dq}{2\pi}$ → integral taken over path for fixed, given E, λ!

∴ → I const, as λ varies
∴ I *adiabatic invariant*

→ in general (including higher dimensions)

$$I_c = \oint_\gamma \underline{p} \cdot d\underline{q} = \iint_\sigma d\underline{p} \wedge d\underline{q}$$ {Liouville Thm, again}

is Poincares relative integral invariant (γ closed curve, enclosing σ)

I_c is *exact* invariant

so $I = I_c \Big|_{E,\ \lambda \text{ constant}}$

is approximation to Poincare invariant

for $\dot{\lambda}/\lambda < \omega_0$.

Now, adiabatic invariant

$$I = \oint_{\lambda, E \text{ fixed}} dq\, p/2\pi \qquad \rightarrow \text{ what is it?}$$

so $I = I(E)$

$$= \oint \frac{p\, dq}{2\pi}$$

$$2\pi \frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E} dq = \oint \frac{dq}{\partial H/\partial p} = \tau$$

$$\therefore \boxed{\partial I/\partial E = 1/\omega}$$

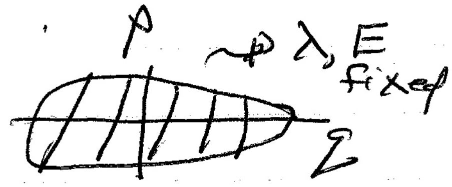
$$\boxed{\partial E/\partial I = \omega}$$

Now, of course:

adiabatic invariant has geometrical significance.

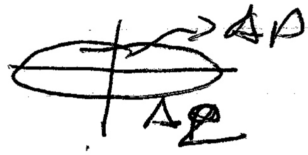
$$I = \oint_{E,\lambda} \frac{p\,dq}{2\pi} = \iint_{E,\lambda} \frac{dp\,dq}{2\pi}$$

i.e.



∴ I corresponds to enclosed area!

e.g. $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = E$



$$\Delta p = (2mE)^{1/2}$$
$$\Delta q = (2E/m\omega^2)^{1/2}$$

$$\text{Area} = \pi\,\Delta q\,\Delta p = 2\pi E/\omega$$

$$\boxed{I = E/\omega}$$

→ for oscillator, adiabatic invariant is <u>action</u>, E/ω.

∴ $\dot{\ell}/\ell < \omega_0 \Rightarrow$

$E \sim \omega \sim \sqrt{g/\ell}$

Physics 200A — Mechanics

→ Adiabatic Invariants: Review

→ if $H = H(p, q, \lambda(t))$

($\lambda(t)$ → parametric dependence)

with a) periodic motion, for fixed λ.

b.) $\frac{1}{\lambda}\frac{d\lambda}{dt} < \omega$

($\frac{1}{\lambda}\frac{d\lambda}{dt}$ → rate of change of parameter; ω → motion frequency)

then $I_\lambda = \oint_{c_\lambda} \underline{p} \cdot d\underline{q}$ ≡ action computed at fixed value of λ is adiabatic invariant

→ adiabatic invariant is COM on time scales $\tau > \omega^{-1}$.

→ adiabatic invariant is intrinsically / implicitly referenced to a given time scale. Same system can manifest multiple adiabatic invariants on different time scales.

→ $I = \oint_c \underline{p} \cdot d\underline{q}$ → Poincare-Cartan Invariant
→ exact COM

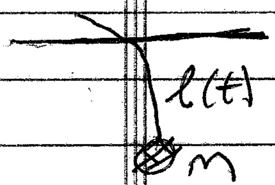
to calculate explicitly, need integrable motion (as in explicit representation of action-angle var.)

but

— $I_\lambda = \oint_{C_\lambda} p \cdot dq$ is approximation to I,

computed for fixed λ. $\dot{I}_\lambda \cong 0$ for $t \gg \omega^{-1}$.

Examples: i) Pendulum — the Prototype



$$\frac{\dot{l}(t)}{l} < \sqrt{g/l}$$

How does Θ vary with l?

$I = E/\omega$, understood $E = \overline{E}$

$\omega = \sqrt{g/l}$

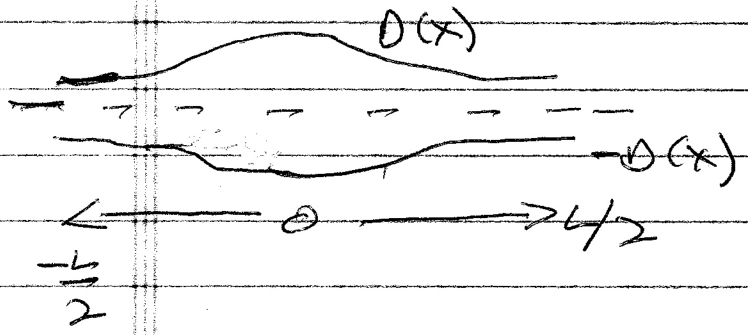
$$\overline{E} = \frac{m}{2} l^2 \overline{\dot{\Theta}^2} + mgl\frac{\overline{\Theta^2}}{2}$$

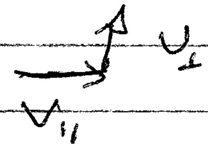
$$= mgl\,\overline{\Theta^2}$$

$$I = m\sqrt{g}\, l^{3/2}\, \overline{\Theta^2}$$

so $\Theta \sim l^{-3/4}$ i.e. amplitude decreases as length increases more generally, $\frac{\theta_{rms}(t)}{\theta_{rms}(0)} \sim \left(\frac{\ell(0)}{\ell(t)}\right)^{3/4}$

2.) Mechanical Mirror





$\tau_{b_\perp} \sim (v_\perp / 2D)^{-1}$ → ⊥ bounce time

$\tau_b \ll L / v_\parallel$

⇓

many bounces (⊥) in time to sense curvature of D.

now,

$$2\pi I = \int_{-D}^{D} m v_\perp \, dy + \int_{D}^{-D} (-m v_\perp) \, dy$$

$$= 4 m D v_\perp$$

$$\boxed{I = \frac{2}{\pi} D m v_\perp}$$ Adiabatic Invariant

$$E = \frac{1}{2} m \left(v_\perp^2 + v_\parallel^2\right)$$

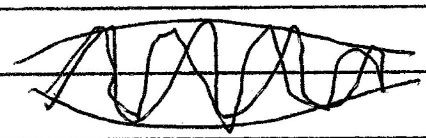
$$\therefore \quad E = \frac{1}{2} m \left(\frac{\pi^2}{4D^2} \frac{I^2}{m^2} + v_{\parallel}^2 \right)$$

$$\Rightarrow \quad \frac{2E}{m} = \frac{\pi^2}{4D^2(x)} \frac{I^2}{m^2} + v_{\parallel}^2$$

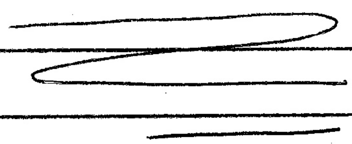
$$\boxed{v_{\parallel}^2 = \frac{2E}{m} - \frac{\pi^2}{4D^2(x)} \frac{I^2}{m^2}}$$

if $\frac{\pi^2 I^2}{4D(x)^2 m^2} > \frac{2E}{m} \Rightarrow$ particle reflected in throat of mirror

i.e.



i.e. on avg.



L

$$I = \frac{2}{\pi} D(x_0)\, m v_{\perp 0}$$

$D(x_0)$ → separation at injection; $m v_{\perp 0}$ → injection momentum

$$\Rightarrow \quad \left(\frac{D(x_0)}{D(x)} \right)^2 v_{\perp 0}^2 > 2E/m$$

$x < L$

∴ for mirroring/trapping:

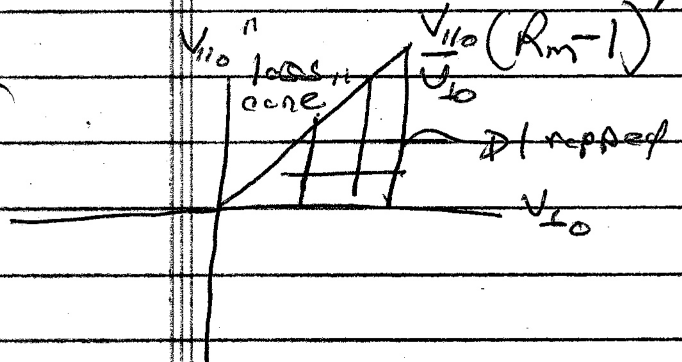
$$\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \left(\frac{D(x_0)}{D(x)}\right)^2 - 1$$

"mirror ratio"

i.e. trapping condition:

$$\boxed{\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \left(\frac{D(x_0)}{D(x)}\right)^2 - 1}$$

R_m



n.b. some class of particles always lost

Now, for trapped/mirroring particles, can determine reflection point x_R from:

$$v_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x_R)^2}\frac{I^2}{m^2} = 0$$

$x_R \leq L/2$

then : for $t > \tau_{b\parallel} = \oint \frac{dx}{|v_\parallel|}$

↓

parallel bounce time for trapped

can have "2^{nd}" adiabatic invariant on time scale $\tau_{b\parallel} >> \tau_{b\perp}$

$$J_\parallel = \oint dx\, p_\parallel$$

$J_\parallel \Rightarrow$ "2^{nd}" ad. inv.
– ∥ bounce inv.

$$\frac{p_\parallel^2}{2m} = E - \frac{\pi^2 I^2}{8 p(x)^2}/m$$

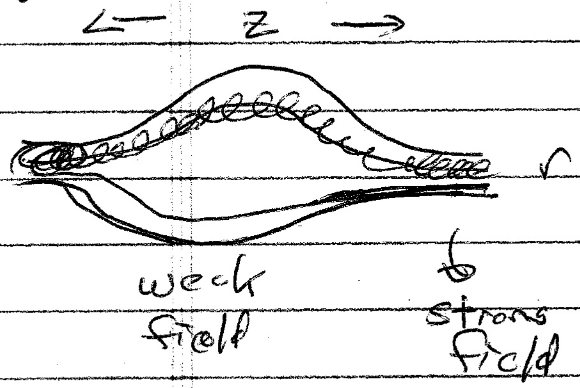
$J_\perp \Rightarrow 1^{st}$ ad inv.
– ⊥ bounce inv.

$$\tau_\perp < \tau < \tau_\parallel < \tau'$$

- $\tau_\perp$ → ⊥ bounce time
- τ → time of $J_\perp$ ad. inv.
- $\tau_\parallel$ → ∥ bounce time
- τ' → time of $J_\parallel$ ad. inv.

N.B. : Can expect 1 adiabatic invariant per closed cyclic orbit (n.b. cyclic orbit a/a action variable sense).

3.) Magnetic Mirror – basis for mechanical mirror



$$\underline{\nabla} \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$

$\neq 0$

Now, consider rate of change of ⊥ Energy

$$\frac{d}{dt}\left(\frac{m v_\perp^2}{2}\right) = q \, \underline{E}_\perp \cdot \underline{v}_\perp$$

avg over 1 cyclotron orbit ⇒

$$d\left(\frac{m v^2}{2}\right) = \int_{\Omega^{-1}} dt \, q \, \underline{E}_\perp \cdot \underline{v}_\perp$$

$\underline{v} \, dt = \underline{l}$

change in energy in 1 cyclotron orbit

$$= \int_{\rho_{\text{gyro circle}}} d\underline{l} \cdot \underline{E}_\perp q = q \oint \underline{E} \cdot d\underline{l}$$

$\rho \to$ gyro-radius

$$= \int d\underline{a} \, q \cdot \nabla \times \underline{E}$$

$$= -\int d\underline{a} \left(\frac{q}{c} \frac{\partial B}{\partial t}\right)$$

$$\cong -\pi \rho^2 \, \frac{q}{c} \frac{\partial B}{\partial t}$$

$\rho^2 = V_\perp^2/\Omega^2$

$\Rightarrow$

$$\delta\left(\frac{mV_\perp^2}{2}\right) \cong -\pi \frac{q}{c} \frac{V_\perp^2}{\frac{q^2 B^2}{m^2 c^2}}$$

$$= -\frac{mV_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but $\delta B = \frac{2\pi}{\Omega} \frac{\partial B}{\partial t}$

δ ↓ change in 1 cyclotron period; ↓ τ_c

$\therefore$

$$\delta\left(\frac{mV_\perp^2}{2}\right) = -\frac{mV_\perp^2}{2} \frac{1}{B} \delta B$$

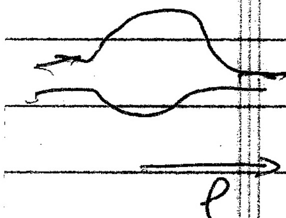
$$\Rightarrow \boxed{\delta\left(\frac{mV_\perp^2}{2B}\right) = 0}$$

so $\boxed{\mu = mV_\perp^2/2B}$ → magnetic moment adiabatic invariant on $t \gg \Omega^{-1}$

Now, for mirroring:

$$\frac{1}{2} m (V_{\parallel}^2 + V_{\perp}^2) = \frac{1}{2} m (V_{\parallel 0}^2 + V_{\perp 0}^2)$$

$$\frac{m V_{\perp}^2(0)}{2B(0)} = \frac{m V_{\perp}^2(\ell)}{2B(\ell)}$$



$$V_{\parallel}^2(0) + V_{\perp}^2(0) = V_{\parallel}^2 + \frac{B(\ell)}{B(0)} V_{\perp}^2(0)$$

$$V_{\perp}^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) = V_{\parallel}^2(\ell) - V_{\parallel}^2(0)$$

for confinement: $V_{\parallel}^2(\ell) = 0 \Rightarrow$

so

$$\boxed{\frac{V_{\parallel}^2(0)}{V_{\perp}^2(0)} < \frac{B(\ell)}{B(0)} - 1}$$

($\frac{B(\ell)}{B(0)}$: mirror ratio)

obvious analogy to:

$$\frac{V_{\parallel 0}^2}{V_{\perp 0}^2} < \frac{\Delta(x_0)}{\Delta(x_R)} - 1$$

$B(\ell) \leftrightarrow 1/\Delta(x_R) \rightarrow$ strong $B \rightarrow$ frequent gyration, frequent bouncing

$B(0) \leftrightarrow 1/\Delta(x_0) \rightarrow$ weak $B \rightarrow$ less frequent bouncing, gyration

Similarly, can define bounce invariant:

$$J_{\parallel} = \oint dl \left[2m\left(E - \mu B(l)\right)\right]^{1/2}$$

i.e. $V_{\parallel}^2(l) = V_{\parallel}^2(0) + V_{\perp}^2(0) - \mu B(l)$

etc.

N.B.: Treatment of adiabatic invariants given here corresponds to lowest order p.t. in $\frac{1}{\lambda}\frac{d\lambda}{dt}\Big/\omega < 1$

"ϵ"

"$O(\epsilon)$" here.