Adiabatic Theory and Wave Action Density

Wave Adiabatic Theory / Wave Kinetics

- frequently encounter problems with slowly varying parameters ⇒ adiabatic theory

⇒

- wave kinetic equation (consequence of Liouville Thm.)

$$\partial_t N + (\underline{v}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \partial_{\underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \partial_{\underline{k}} N$$

$$= C(N)$$; obvious analogy to Boltzmann Eqn.

$$\underset{\frown}{N} \equiv \mathcal{E} / \omega_{\underline{k}}$$ ≡ wave action density / wave quanta density

$\mathcal{E}$ ↓ wave energy density

$$\mathcal{E} = \frac{\partial}{\partial \omega} (\omega \epsilon_{\underline{k}}) \Big|_{\omega_{\underline{k}}} \frac{|E_{\underline{k}}|^2}{8\pi}$$, for e.s. waves

Characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} \hat{k} + \underline{V} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\underset{\uparrow}{\omega} + \overset{\downarrow}{\underline{k} \cdot \underline{V}})$$

refraction by shear

refraction by parametric variation

- need:

$$\underset{\frown}{\omega < \frac{1}{\lambda} \frac{d\lambda}{dt}}$$ λ ≡ parameter

$$\boxed{\text{space and time scale separation}}$$

$$\frac{1}{N} (\underline{v}_{gr} \cdot \nabla N) \ll \omega \Rightarrow \underline{k} \cdot \underline{v}_{gr} < \omega$$

$O(N)$ → interactions with comparable scale.

<u>Examples</u>:

- linear theory of Langmuir turbulence
  i.e. when will phonon grow?
- QL theory of Langmuir turbulence
  i.e. determine evolution of plasma
  energy → net impact?!
- drift waves and sheared flow.

⋮

$N = \frac{\varepsilon}{\omega}$

→ dynamics?!

Fundamentals of Wave Kinetics

→ where does conservation of action emerge from?

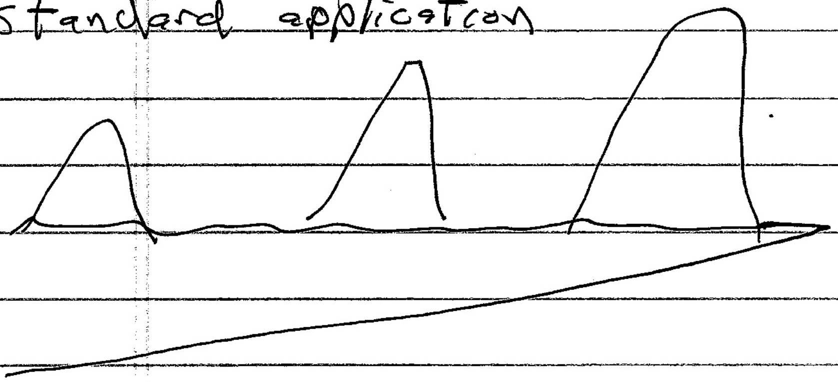
→ answer: <u>phase</u> <u>symmetry</u> (of wave train) underlies wave kinetics

→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves"
Chapt. 14.

⇒ standard application

⇒



beach ↔

{ waves in shallow water

- influx of wave energy

- depth $H(x,y)$ decreases

⇒ wave amplification, breaking.

- Derivation

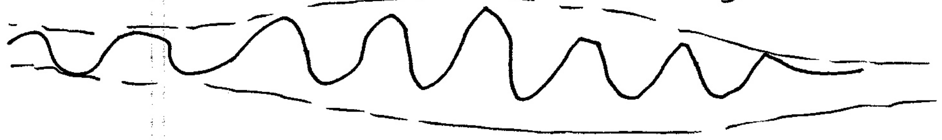
Consider a system, [like ideal fluid MHD, acoustics] which can ~~be~~ be described in terms of displacement $\underline{\xi}$:

i.e. $$\underline{\xi} = \text{re}\left\{ A e^{i\overset{\text{phase}}{\phi}} + A^* e^{-i\phi} \right\}$$

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x}\, \mathcal{L}(\xi)$$

-Envision a wave train, with slowly varying amplitude, so eikonal approach optimal i.e. fast variation in phase, a la WKB :



$$S = \int dt \int d\underline{x}\, \mathcal{L}(\omega, \underline{k}, \underset{\underline{\text{amplitude}}}{a}) \qquad \underline{k} = \underline{\nabla}\phi$$

$$\omega = -\partial_t \phi$$

$$= \int dt \int d\underline{x}\, \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

-neglect all corrections to eikonal theory.

→ here L corresponds to period-averaged Lagrangian
- ϕ undetermined to const → phase symmetry!

∴ to vary:

$$\delta S/\delta a = 0$$

$$\delta S/\delta \phi = 0$$

[illegible]

↓

$$\left[G(k,\omega) \Leftrightarrow \underline{\epsilon} \right]$$

Now, in linear theory:

$$\mathcal{L} = G(\omega, k)\, a^2$$

$G(\omega, \underline{k}) = 0$ dispn
$\omega^2 = k^2 c_s^2$,

i.e. for MHD, as in wave section:

$$\mathcal{L} = \frac{1}{2}\rho\dot{\xi}^2 - \frac{1}{2}\rho\left[\underline{\underline{D}}(k, x, t)\right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

↳ eikonal form of stiffness matrix (→ potential energy)

$$\Rightarrow \underline{\xi} \cdot \underline{\underline{M}} \cdot \underline{\xi}$$

↓

$M(k, \omega, \theta)$, as for linear waves

If: $\underline{\xi} = \underline{A}e^{i\phi} + A^* e^{-i\phi}$

$\underline{\underline{i}}$ $$G(\omega,\underline{k}) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi,\underline{x},t)\right]^2\right]$$

Now, 1) $\delta S/\delta a = 0$

$$\Rightarrow \quad G(\omega,k) = 0$$

but

$$G(\omega,k) = \rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi,\underline{x},t)\right]^2$$

$$= \rho\omega^2 - \underset{\hookrightarrow \text{ stiffness fctn.}}{D^2}$$

$\Rightarrow$ dispn. relation

2) $\delta S/\delta\phi = 0$

$$\delta S = \int dt\int d^3x\left\{\frac{\partial\mathcal{L}}{\partial(-\phi_t)}\delta(-\phi_t) + \frac{\partial\mathcal{L}}{\partial(\phi_{\underline{x}})}\delta(\phi_{\underline{x}})\right\}$$

end pts fixed, ibp

$$= \int dx\int d^3x\left\{\partial_t\left(\frac{\partial\mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial(\underline{\nabla}\phi)}\right)\right\}\delta\phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t\left(\frac{\partial\mathcal{L}}{+\partial(-\phi_t)}\right) - \underline{\nabla}\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0$$

Now, have: $G(k, \omega) = 0$ (dispn. reln.)

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega}\, d\omega + \frac{\partial G}{\partial \underline{k}}\cdot d\underline{k} = 0$$

$$\therefore \quad v_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}$$ (taking ω)

$\therefore$

$$\partial_t\left((\partial G/\partial \omega)\, a^2\right) + \nabla\cdot\left[\frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}\,\frac{\partial G}{\partial \omega}\, a^2\right] = 0$$

and so $N \equiv \frac{\partial G}{\partial \omega}\, a^2$

$$\frac{\partial N}{\partial t} + \nabla\cdot\left(\underline{v}_{gr} N\right) = 0$$

(N not <u>yet</u> action)

Also note energy is conserved $\Leftrightarrow$ G invariant to time translations.

so, Noethers thm $\Rightarrow$ there exists an [redacted] energy conservation equation

have $\quad \mathcal{L} = G(k,\omega)\, a^2$

$$\partial \mathcal{L}/\partial a = 0 \quad \Rightarrow \quad G(\omega, k) = 0$$

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

and of course:

$$\underline{\nabla} \times \underline{k} = 0 \ , \quad \text{as} \quad \underline{k} = \underline{\nabla} \phi$$

$$\frac{\partial \underline{k}}{\partial t} = - \frac{\partial \omega}{\partial \underline{x}} \ , \text{as} \quad \partial_t \underline{\nabla} \phi = - \underline{\nabla} \left(- \frac{\partial}{\partial t} \phi \right)$$ ✓

Now, $\mathcal{L} = 0$, as $G(k,\omega) = 0$

as expect $\quad \dfrac{\partial \mathcal{L}}{\partial \omega} \Rightarrow N, \qquad \omega \dfrac{\partial \mathcal{L}}{\partial \omega} \Rightarrow \mathcal{E}$

$\Rightarrow 0$, [redacted] creatively

$$\rightarrow \frac{\partial}{\partial t} \left(\omega \frac{\partial \mathcal{L}}{\partial \omega} - [redacted]\ \mathcal{L} \right) + \nabla \cdot \left[- \omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right] = 0$$

$$-\frac{\partial G / \partial k}{\partial G / \partial \omega} \frac{\partial G}{\partial \omega} a^2$$

[redacted] $v_{\underline{g}r}\, \mathcal{E}$.

$$\partial_t \left(\omega \mathcal{L}_\omega + \mathcal{L} \right) + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

check:

$$(\partial_t \omega) \mathcal{L}_\omega + \omega \, \partial_t (\mathcal{L}_\omega) - \partial \mathcal{L} / \partial t$$

$$+ \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

$\Rightarrow$ but $\partial_t \mathcal{L}_\omega = \nabla \cdot (\mathcal{L}_{\underline{k}})$

$$\therefore \ (\mathcal{L}_\omega)(\partial_t \omega) + \omega \, \cancel{\nabla \cdot (\mathcal{L}_k)} - \omega \left(\cancel{\nabla \cdot \mathcal{L}_{\underline{k}}} \right)$$

$$- \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) \cdot \underline{\nabla} \omega - \frac{\partial \mathcal{L}}{\partial t}$$

but $\partial_t \underline{k} = - \underline{\nabla} \omega$

$$(\partial_t \omega)(\mathcal{L}_\omega) + (\partial_t \underline{k}) \cdot \frac{\partial \mathcal{L}}{\partial \underline{k}} - \frac{\partial \mathcal{L}}{\partial t} = 0 \quad \checkmark$$

(identity)

$$\Rightarrow \boxed{\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0}$$

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

so

$$\mathcal{E} \equiv \omega \frac{\partial \mathcal{L}}{\omega} \rightarrow \overset{\text{wave}}{\text{energy}} \text{ density}$$

so

$$\underset{\equiv N}{\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega} \Rightarrow \overset{\text{wave}}{\text{action}} \text{ density } !$$

so have:

$$\boxed{\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0}$$

wave - kinetics

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \underline{\nabla} N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\partial \omega / \partial \underline{x} \, N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$
(i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{v}_{gr} N \right] = 0$$

Observe:

→ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

→ continuity-type equation in $\underline{x}$-space, for packet

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot (\underline{v}_{gr} N) = 0$$

Also observe:

- seeming issue re:

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

<u>Now</u> $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian)
(partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian)
(total) relation, <u>following</u> packet)

(here $\omega = D(\underline{k}, \underline{x}, t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr}\cdot\nabla\underline{k}$$

$$= -\frac{\partial\omega}{\partial\underline{x}} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\underline{k}}{\partial\underline{x}}$$

∴

$$\frac{\partial\underline{k}}{\partial t} = -\frac{\partial D}{\partial\underline{x}}$$ agree!

→ Now, can convert from N to E !

<u>i.e.</u> $N = \mathcal{E}/\omega$

$$\left.\frac{dN}{dt}\right|_{rays} = \frac{d}{dt}\left(\mathcal{E}/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\Big|_{rays} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\Big|_{rays} = 0$$

Now $\frac{d\omega}{dt} = \partial_t\omega + \frac{\partial\omega}{\partial\underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{d\underline{k}}{dt}$

From eikonal eqns:

$$= \partial_t\omega + \cancel{\frac{\partial\omega}{\partial\underline{x}}\cdot\frac{\partial\omega}{\partial\underline{k}}} - \cancel{\frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\omega}{\partial\underline{x}}}$$

so if $\partial_t\omega = 0$

$\therefore \quad \frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0.$

so $\quad \partial_t\mathcal{E} + \underline{v}_{gr}\cdot\underline{\nabla}\mathcal{E} - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\mathcal{E} = 0$

and exploiting Liouville Thm, etc ⇒

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t\mathcal{E} + \nabla\cdot\left[\underline{v}_{gr}\,\mathcal{E}\right] = 0}$$

so, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot \left[\underline{v}_{gr} \mathcal{E} \right] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

$$\Rightarrow \boxed{\underline{\nabla} \cdot \left[\underset{\rightarrow}{v}_{gr} \mathcal{E} \right] = 0}$$

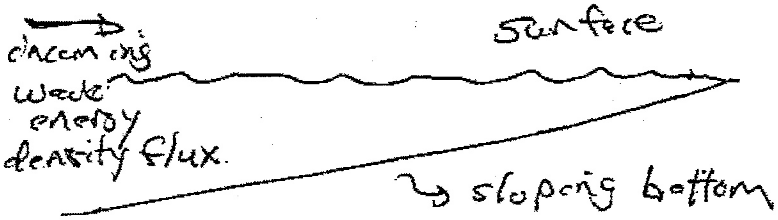
incompressible wave energy flux !

$\Rightarrow v_{gr}$ drops $\Rightarrow$

$\mathcal{E} \uparrow \Rightarrow$ blocking, breaking

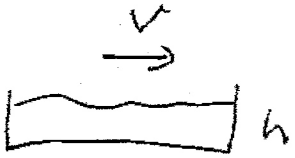
③ The beach....

Consider:



$H = H(x)$

Now, in shallow water
($\lambda > H$)



$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0$$

slope
b

shallow water eqns.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}$$

$v = \cancel{v_0} + \tilde{v}$, $h = H + \tilde{h}$

$\Rightarrow$ $-i\omega\tilde{h} + ikH\tilde{v} = 0$
$-i\omega\tilde{v} = -ikg\tilde{h}$

$\therefore \rightarrow \omega^2 = k^2 g H$ is dispersion relation

→ analogy with acoustics is obvious

$h \Leftrightarrow \rho$ $\quad c_s^2 = gH$
$v \Leftrightarrow v$ $\quad$ etc.

energy

$$\frac{\partial \tilde{v}}{\partial t} = -g \frac{\partial \tilde{h}}{\partial x} \qquad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H \frac{\partial \tilde{v}}{\partial x} \qquad (2)$$

$$\Rightarrow \quad (1) \times \tilde{v} \quad + \quad (2) \times \left(g \frac{\tilde{h}}{H} \right)$$

$$\therefore \quad \frac{\partial}{\partial t} \tilde{v}^2 = -g \, \tilde{v} \frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H} \frac{\partial}{\partial t} \tilde{h}^2 = -g \frac{\cancel{H}}{\cancel{H}} \tilde{h} \frac{\partial \tilde{v}}{\partial x}$$

$$\therefore \quad \frac{\partial}{\partial t} \left(\frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H} \right) + \frac{\partial}{\partial x} \left(g \tilde{h} \tilde{v} \right) = 0$$

is energy theorom

$$\Rightarrow \quad \mathcal{E} = \frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H}$$ is wave energy density

$$\omega / k = (gH)^{1/2}$$ is wave phase velocity

so .... as no explicit time dependence:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(V_{gr} \mathcal{E} \right) = 0$$

$$\Rightarrow \quad V_g(x)\,\mathcal{E}(x) = V_\infty\,\mathcal{E}_\infty = \underset{\substack{\downarrow \\ \text{incoming} \\ \text{wave flux}}}{I}$$

$V_g = \sqrt{gH(x)}$

↳ shallow water waves have zero dispersion

$$\therefore \quad \sqrt{gH(x)}\;\mathcal{E}(x) = I$$

as $x \rightarrow$ shore $V_{gr} \downarrow$ so wave energy ~~[illegible]~~ must increase.

$$\text{Now} \quad \mathcal{E}(x) = \frac{\overline{\tilde{v}^2}}{2} + \frac{g\overline{\tilde{h}^2}}{2H} \simeq \frac{g\tilde{h}^2}{2H}$$

$$\sqrt{gH(x)}\;\frac{g\tilde{h}^2}{2H(x)} = I$$

$$\frac{\tilde{h}^2}{H(x)^2} = \frac{2I}{(\sqrt{g})^3}\left(\sqrt{H(x)}\right)^{-3}$$

$$\text{i.e.} \quad \boxed{(\tilde{h}/H)^2 \sim (\text{const})\, I \,/\, (H(x))^{3/2}}$$

i.e. $\tilde{h}/H \rightarrow 1 \Leftrightarrow$ breaking $\Leftarrow$ as $H(x)$ drops.

N.B.:

→ if know bottom profile, can deduce displacement profile, and approximate breaking point.

→ 2D bottom contours ⇒ wave refraction

$$\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}} = -kg\left(\frac{\partial}{\partial \underline{x}} H(x,y)\right)$$

i.e. wavefronts tend to align with bottom contours approaching shore.