Fluids: Introduction and Potential Flow

4.) Euler Equations and Ideal Fluid Mechanics { Basic Eqns.

ideal – 'dry' H_2O
– no viscosity, molecular diffn.

i.) Euler Eqns.

a.) Continuity Equation

Consider { volume V of fluid, density $\rho(\underline{x},t)$

Then, mass conservation $\Rightarrow$

$$\frac{dM}{dt} = \underbrace{\frac{\partial}{\partial t}\int dV \rho(\underline{x},t)}_{\text{net change in mass}} = \underbrace{-\int d\underline{S}\cdot\underbrace{[\rho(\underline{x},t)\,\underline{V}(\underline{x},t)]}_{\text{mass density flux}}}_{\text{net out/in flow}}$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\int dV \rho(\underline{x},t) = -\int dV \,\underline{\nabla}\cdot(\rho \underline{V})$$

$\therefore$

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{V}) = 0}$$

Continuity Eqn.
(nonlinear)

$\rho \underline{V} \equiv$ mass density flux

$$\underbrace{\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho}_{\text{advection}} = \underbrace{-\rho\, \underline{\nabla} \cdot \underline{v}}_{\text{compression}}$$

b.) Momentum Balance Equation

Consider force acting on fluid element of density $\rho(\underline{x}, t)$ and velocity $\underline{v}(\underline{x}, t)$

i.e. $\rho \underline{a} = \underline{f}$ → force density

$\underline{f} = -\underline{\nabla} p + \underline{f}_{ext}$

$-\underline{\nabla} p$: pressure gradient (fluid on self)

$\underline{f}_{ext}$ → external force, i.e. $\underline{f}_{ext} = \underline{g} \rho$

As to $\underline{a}$, $\underline{v} = \underline{v}(\underline{x}, t)$, so need consider:

- change in velocity (i.e. acceleration) of fixed point
  i.e. $\underline{a} = \frac{\partial \underline{v}}{\partial t}$
- rate of change of momentum of moving fluid particle (i.e. moving in space)

then

$$d\underline{v} = \left(\frac{\partial \underline{v}}{\partial t}\right) dt + \underline{dr} \cdot \underline{\nabla}\, \underline{v}$$

local acceleration

particle moves in spatially dependent velocity field.

or

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}$$

convective, advective, substantive derivative

as

$$\frac{d\underline{x}}{dt} = \underline{v}\,(\underline{x}, t)$$

$$\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = -\underline{\nabla}p + \underline{f}$$

{ Euler Eqn. (momentum balance, nonlinear)

Useful to note for momentum flux;

$\rho\underline{V} \equiv$ fluid momentum-per-volume { Combine continuity, Euler

$$\therefore\ \frac{\partial}{\partial t}(\rho\underline{V}) = \underline{V}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\underline{V}}{\partial t}$$

$$= -\underline{V}\left(\rho(\underline{\nabla}\cdot\underline{V}) + \underline{V}\cdot\underline{\nabla}\rho\right) + \rho\left(-\underline{V}\cdot\underline{\nabla}\,\underline{V} - \frac{\underline{\nabla}p}{\rho}\right)$$

$$= -\left(\rho\left[\underline{V}(\underline{\nabla}\cdot\underline{V}) + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] + \underline{V}(\underline{V}\cdot\underline{\nabla}\rho)\right) - \underline{\nabla}p$$

$$= -\underline{\nabla}\cdot\left(\rho\,\underline{V}\,\underline{V} + \underline{\underline{I}}\,p\right)$$

→ momentum flux tensor

→ pressure

identity tensor

Reynolds stress tensor

(analogous Maxwell stress tensor)

i.e.

$$\frac{\partial}{\partial t}\rho V_i = -\frac{\partial}{\partial x_k}\Pi_{ik}$$

dynamic pressure (thermal) pressure

$$\boxed{\Pi_{ik} = \rho V_i V_k + \delta_{ik} p}$$ → Defines momentum flux

i.e. $\rho \underline{V} \equiv$ momentum density

$\Rightarrow$

$$\frac{dP}{dt} = \frac{\partial}{\partial t}\int dV \, \rho \underline{V} = -\int dV \, \underline{\nabla} \cdot \left(\rho \underline{V}\underline{V} + \underline{\underline{I}} p \right)$$

$$= -\int d\underline{S} \cdot \left(\rho \underline{V}\underline{V} + \underline{\underline{I}} p \right)$$

defines momentum outflow

$\therefore$ $\Pi_{ik} \, dS_k \equiv$ momentum flux in i^{th} direction

c.) Energy Equation - Ideal Fluid

In ideal fluid, no heat is exchanged between fluid elements $\Rightarrow$ motion adiabatic

→ beware terminology

$S \equiv$ entropy per mass

$\Rightarrow$ $\frac{dS}{dt} = 0$

$$\boxed{\frac{\partial S}{\partial t} + \underline{V} \cdot \underline{\nabla} S = 0}$$ adiabatic eqn for fluid.

as $$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial s}{\partial t} + \underline{v} \cdot \underline{\nabla} s = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho s) + \underline{\nabla} \cdot (\rho s \underline{v}) = 0}$$ (Eqn. Continuity for entropy)

$\rho s \underline{v} \equiv$ entropy flux

ii) Applications – Euler Eqns.

Now, furthur develop a.) – Bernoulli Eqn.
b.) – Energy Flux
c.) – Kelvin Thm
} topics in ideal fluids

a.) Bernoulli Eqn.

Useful to note for isentropic fluid (constant entropy)

$\Rightarrow$ s = const.

but $$dw = Tds + Vdp$$

dw: specific enthalpy

V: $\sim 1/\rho \equiv$ specific volume

$dE = dQ - p\,dv$
$= Tds - p\,dV$
$dw = Tds + V\,dp$

i.e. $dE = dQ - p\,dV$
$= T\,dS - p\,dV$

$$\underbrace{W}_{\text{enthalpy}} = E + PV$$

(enthalpy is Legendre transform of energy)

Const. entropy $\Rightarrow$ $dw = V\,dp$
$= \frac{dp}{\rho}$

$\Rightarrow$ can re-write Euler equation as:

$$\boxed{\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\, W}$$

{ Euler Eqn. - Isentropic fluid

Now, $\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{V}\times(\underline{\nabla}\times\underline{V}) + \frac{1}{2}\nabla(V^2)$

$$\boxed{\underline{\nabla}\times\underline{V} = \underline{\omega} \equiv \text{vorticity}}$$

$\Rightarrow$

$$\frac{\partial \underline{V}}{\partial t} = \underline{V}\times\underline{\omega} - \underline{\nabla}\left(W + \frac{V^2}{2}\right)$$

i.e. $\rho\frac{\partial \underline{v}}{\partial t} = -\underline{\nabla}\left(W + \frac{V^2}{2}\right) + \underline{V}\times\underline{\omega}$

Magnus force (analogy $\underline{J}\times\underline{B}$)

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla}\times(\underline{V}\times\underline{\omega})}$$

Vorticity induction eqn.

$$= -\underline{V}\cdot\underline{\nabla}\,\omega + \underline{\omega}\cdot\underline{\nabla}\,\underline{V} - \underline{\omega}\,\underline{\nabla}\cdot\underline{V}$$

→ what is vorticity ?

- $\underline{\omega} = \underline{\nabla} \times \underline{V}$
  → describes rotation of fluid element
  → 2 ⊗ effective local angular velocity of fluid
  i.e. $\delta \underline{V} = [\underline{\omega} \times \underline{r}]/2$

  → obvious analogy $\underline{B}, \underline{J} \Leftrightarrow \underline{V}, \underline{\omega}$

- vorticity is critical as it controls the momentum evolution not governed by Bernoulli's law. This is generally the non-trivial part (i.e. roll-up, turbulence, etc.)

so $$\boxed{\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V} - \underline{\omega}\,\underline{\nabla}\cdot\underline{V}}$$ Vorticity Evolution Equation $\underline{\nabla}\cdot\underline{V} = 0$

if compressible: $\boxed{\frac{d}{dt}\frac{\underline{\omega}}{\rho} = \frac{\underline{\omega}\cdot\nabla\underline{V}}{\rho}}$ → $\underline{\omega}/\rho$ 'frozen into' flow

Now, useful to define:

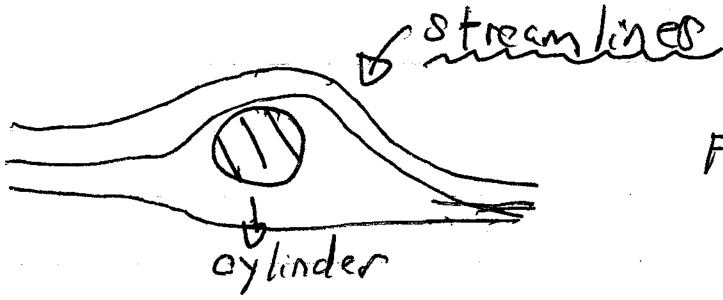
- stationary flow $\partial\underline{V}/\partial t = 0$

- streamline ≡ lines s/t tangent to streamline gives fluid velocity (i.e. F①)

i.e. parametrize streamline via:

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$

e.g.



F①

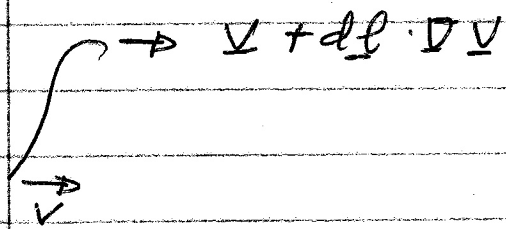
For stationary flow:

$$0 = \underline{V}\times\underline{\omega} - \nabla\left(W + \frac{\underline{V}^2}{2}\right)$$

frozen-in → vector

ξ

Consider line ℓ moving with fluid particles. $d\underline{\ell}$ is segment



$$\frac{d}{dt}\underline{\ell}_2 = \underline{v}(\underline{x}+d\underline{\ell},t)$$

$$\frac{d}{dt}\underline{\ell}_1 = \underline{v}(\underline{x},t)$$

in dt $\quad \Delta d\underline{\ell} = dt\, d\underline{\ell}\cdot\nabla\underline{V}$

so $\quad \dfrac{d\,d\underline{\ell}}{dt} = d\underline{\ell}\cdot\nabla\underline{V}$

but $\underline{\omega}/\rho$ obeys same eqn. as $d\underline{\ell}$

∴ $\boxed{\underline{\omega}/\rho \text{ 'frozen into' fluid}}$

Now, $\underline{\ell} \equiv$ unit vector tangent to streamline

$$0 = \cancel{\underline{\ell}\cdot(\underline{V}\times\underline{\omega})} - \underline{\ell}\cdot\underline{\nabla}\left(W+\frac{V^2}{2}\right)$$

$\underline{\ell} = \underline{V}/|\underline{V}|$

$$\Rightarrow \quad \frac{\partial}{\partial \ell}\left(W+\frac{V^2}{2}\right) = 0$$

statement of energy balance.

$$\Rightarrow \quad \boxed{W + \frac{V^2}{2} = \text{const.}}$$

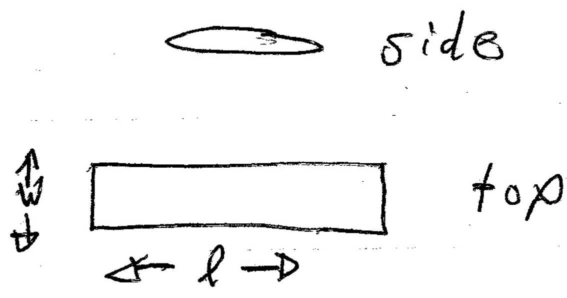
static Bernoulli's Eqn., const. diffn for each streamline

Note if gravitational force,

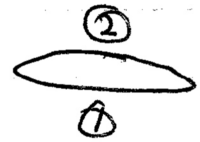
$$W + \frac{V^2}{2} + gZ = \text{const.}$$

Application : Lift Theorem

Consider airplane wing ($\ell \gg w$)



$\therefore \quad \frac{P}{\rho} + \frac{V^2}{2} = \text{const.}$



$\Rightarrow \quad \frac{P_1}{\rho} + \frac{V_1^2}{2} = \frac{P_2}{\rho} + \frac{V_2^2}{2}$ (infinitesimally thin wing)

$\therefore \quad \frac{P_1 - P_2}{\rho} = \frac{V_2^2}{2} - \frac{V_1^2}{2}$

$= \frac{1}{2}(V_2 - V_1)(V_2 + V_1)$

For infinitesimal wing: $\frac{V_1 + V_2}{2} \cong V_p$ speed of plane

$V_2 - V_1 = \frac{1}{W} \oint_{wing} \underline{v} \cdot d\underline{\ell}$

defines circulation

$\equiv \frac{\Gamma}{W}$ (circulation)

$\Rightarrow \quad \frac{P_1 - P_2}{\rho} = \frac{\Gamma}{W}$

$\therefore \quad F_{Lift} = (P_1 - P_2) A_{wing}$

$= \frac{\rho \Gamma}{W} (\ell W) V_p = \rho \Gamma \ell V_p$

$\therefore$ $\boxed{F_{lift} = \rho \Gamma l V_p}$ lift theorem

$$\boxed{\Gamma \equiv \oint_{wing} \underline{v} \cdot d\underline{l} \quad \leftrightarrow \text{circulation}}$$

Note: – high air temp. $\Rightarrow$ low ρ $\therefore$ take-off difficult (Phoenix, summer '90)

– can empirically, theoretically relate

$$\Gamma = \underbrace{C_L}_{\text{lift coefficient } (\measuredangle \text{ attack; shape; etc.})} V_p W$$

$$\Rightarrow \quad F_L = \rho A_{wing} C_L V_p^2$$

More generally:

Drag Force $\equiv$ Force on object in direction of motion ($\to$ drag coefficient)

Lift Force $\equiv$ Force on object $\perp$ direction of motion ($\to$ lift coefficient)

D.) Kelvin's Theorem

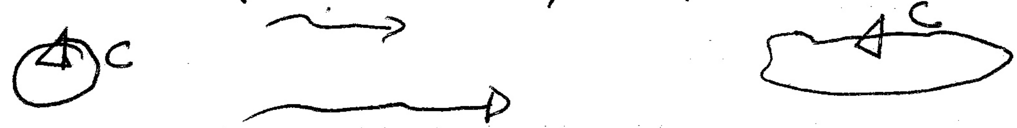
$$\Gamma = \oint_C \underline{v} \cdot d\underline{l} \quad \equiv \text{defines circulation about contour } C$$

{ obvious analogue → Kelvin's Theorem

Consider evolution of circulation about contour in dynamic flow

(deformation by flow)

i.e.



where: - contour not broken (i.e. ideal fluid)
- C still encloses **same** fluid element

Thus, need (consider) $\frac{d}{dt}\oint_C \underline{v}\cdot d\underline{\ell} = d\Gamma/dt$

where - both contour and flow evolve
⇓
- total derivative

For contour element $d\underline{\ell} = d(\underline{r}_2-\underline{r}_1)\Big|_{\lim \underline{r}_2\to\underline{r}_1} = \delta\underline{r}$

so

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot\frac{d\,\delta\underline{r}}{dt}$$

$$= \oint d\underline{\ell}\cdot\frac{d\underline{v}}{dt} + \cancel{\oint \underline{v}\cdot\delta\underline{v}} \rightarrow \text{closed contour}$$

$$\Rightarrow \quad \frac{d\Gamma}{dt} = \oint d\underline{\ell} \cdot \frac{d\underline{v}}{dt}$$

but for ideal fluid, $\frac{d\underline{v}}{dt} = -\underline{\nabla} w$

(ideal $\Leftrightarrow$ isentropic)

$$\frac{d\Gamma}{dt} = -\oint d\underline{\ell} \cdot \underline{\nabla} w = 0$$

$\therefore$ **Kelvin's Thm:** In an ideal (isentropic) fluid, $\Gamma = \oint_C \underline{v} \cdot d\underline{\ell}$ (circulation) is conserved

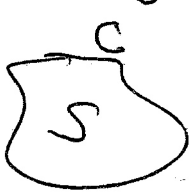
n.b. obvious analogy is Poincaré–Cartan invariant $I_p = \oint \underline{p} \cdot d\underline{q}$. $dI_p/dt = 0$

useful also to note:

$$\Gamma = \oint_C \underline{v} \cdot d\underline{\ell} = \int_S d\underline{s} \cdot \underline{\omega}$$

$\underline{\omega} = \underline{\nabla} \times \underline{v}$ (vorticity)

S enclosed by C



$2\pi r V_{rot} = \pi r^2 \omega$

$V_{rot} = r\Omega_{rot}$ $\quad \Omega_{rot} = \omega/2$

i.e. circulation = flux of vorticity thru surface enclosed by contour C.

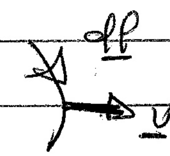
$\rightarrow$ Magnetic Analogy, Alfven's Thm:

$$\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = 0$$

$$-\frac{1}{c}\frac{\partial B}{\partial t} = \underline{\nabla}\times\underline{E}\,, \qquad \underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \cancel{\eta J}\ 0$$

Now,

$$\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = \int \frac{\partial \underline{B}}{\partial t}\cdot d\underline{a} + \int \underline{B}\cdot\frac{d\underline{a}}{dt}$$



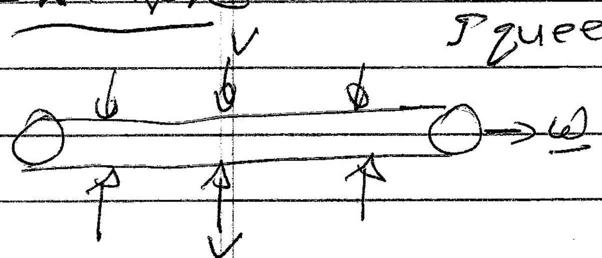
$$= -c\int \underline{\nabla}\times\underline{E}\cdot d\underline{a} + \int \underline{B}\cdot(\underline{v}\times d\underline{\ell})$$

$$= -c\int \underline{E}\cdot d\underline{\ell} - \int d\underline{\ell}\cdot(\underline{v}\times\underline{B})$$

$$= -c\int \underline{E}\cdot d\underline{\ell} + c\int \underline{E}\cdot d\ell \quad = 0 \ \checkmark$$

Example:

Squeezed at $\underline{v} = v(r,t)\,\hat{r}$



compressed vortex tube.
(cohv. $\underline{\nabla}\cdot\underline{v} \neq 0$)

How does vorticity evolve?

$\underline{\omega} \cdot \underline{\nabla}\, \underline{v} = 0$

$\frac{\omega}{\rho} A \sim$ const. (Kelvin)

$\rho A \sim$ cont (Mass)

7)

$\omega/\rho \sim$ const,

$\therefore \quad \omega \sim \rho \sim 1/A$.

c.) Energy Equation and Energy Flux

For energy evolution, need consider:

$$\mathcal{E} = \underbrace{\frac{\rho v^2}{2}}_{\substack{\text{kinetic}\\ \text{energy}\\ \text{density}}} + \underbrace{\rho\epsilon}_{\substack{\text{internal}\\ \text{energy density}\\ \text{(i.e. thermal)}}}$$

As with momentum density, consider $\partial\mathcal{E}/\partial t$

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t}\Big(\overset{①}{\frac{\rho v^2}{2}} + \overset{②}{\rho\epsilon}\Big)$$

①:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho \underline{v}\cdot\frac{\partial \underline{v}}{\partial t}$$

$$= \underbrace{-\frac{1}{2}v^2\,\underline{\nabla}\cdot(\rho\underline{v})}_{\substack{\text{from eqn.}\\ \text{Continuity}}} \underbrace{-\underline{v}\cdot\underline{\nabla}p - \rho\underline{v}\cdot(\underline{v}\cdot\underline{\nabla}\,\underline{v})}_{\text{from momentum balance}}$$

now $\quad \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\underline{v}\times\underline{\omega} + \frac{1}{2}\underline{\nabla}v^2$

$\Rightarrow$

$$\rho \underline{V} \cdot \underline{V} \cdot \nabla \underline{V} = \rho \underline{V} \cdot \left(-\underline{V} \times \underline{\omega} + \tfrac{1}{2} \nabla V^2 \right)$$

$$= \rho \underline{V} \cdot \nabla \frac{V^2}{2}$$

Also $\quad dw = T ds + V dp$

$$= T ds + \frac{dp}{\rho}$$

$$\Rightarrow \quad \nabla p = \rho \nabla w - \rho T \nabla s$$

$$\therefore \quad \frac{\partial}{\partial t} \left(\frac{\rho V^2}{2} \right) = -\frac{V^2}{2} \nabla \cdot (\rho \underline{V}) - \rho \underline{V} \cdot \nabla \left(\frac{V^2}{2} + w \right)$$

$$+ \rho T \underline{V} \cdot \nabla s$$

② $\quad \dfrac{\partial}{\partial t} (\rho \epsilon)$

Useful to transform using thermodynamic identity

$$d\epsilon = dQ - p dV$$

$$= T ds - p dV$$

but $V = 1/\rho$

$$dV = -\frac{d\rho}{\rho^2}$$

$\Rightarrow$

$$d\epsilon = T ds + \frac{p}{\rho^2} d\rho$$

so $$d(\rho\epsilon) = \rho\, d\epsilon + \epsilon\, d\rho$$

$$= \rho\left(T ds + \frac{p}{\rho^2} d\rho\right) + \epsilon\, d\rho$$

$$= \left(\frac{p}{\rho} + \epsilon\right) d\rho + \rho T ds$$

but $$\epsilon + \frac{p}{\rho} = \epsilon + pV = W$$

$\therefore$ $$d(\rho\epsilon) = W\, d\rho + \rho T ds$$

$\Rightarrow$ $$\frac{\partial}{\partial t}(\rho\epsilon) = W\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}$$

$$= -W\,\underbrace{\nabla\cdot(\rho\underline{V})}_{\text{from continuity}} - \rho T\,\underbrace{\underline{V}\cdot\nabla s}_{\text{from adiabatic equation}}$$

Thus, combining ①, ②

$$\frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}+\rho\epsilon\right) = -\frac{V^2}{2}\,\nabla\cdot(\rho \mathbf{V}) - \rho\mathbf{V}\cdot\nabla\left(\frac{V^2}{2}+w\right)$$

$$+\cancel{\rho T\,\mathbf{V}\cdot\nabla s} - w\,\nabla\cdot(\rho\mathbf{V}) - \cancel{\rho T\,\mathbf{V}\cdot\nabla s}$$

$$= -\left(\frac{V^2}{2}+w\right)\nabla\cdot(\rho\mathbf{V}) - \rho\mathbf{V}\cdot\nabla\left(\frac{V^2}{2}+w\right)$$

$$= -\nabla\cdot\left(\rho\mathbf{V}\left(\frac{V^2}{2}+w\right)\right)$$

$$= -\nabla\cdot\left(\mathbf{V}\left(\frac{\rho V^2}{2}+w\right)\right)$$

Thus, have:

$$\frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}+\rho\epsilon\right) + \nabla\cdot\left(\rho\mathbf{V}\left(\frac{V^2}{2}+w\right)\right) = 0$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\int dV\left(\frac{\rho V^2}{2}+\rho\epsilon\right) = -\int dV\,\nabla\cdot\left[\rho\mathbf{V}\left(\frac{V^2}{2}+w\right)\right]$$

change in energy in volume V $= -\oint d\mathbf{S}\cdot\left[\rho\mathbf{V}\left(V^2/2+w\right)\right]$

energy flux thru $\mathbf{S}$, enclosing V

$\therefore$ $\boxed{\text{energy density flux}_{\text{(thermal)}} \equiv \rho \underline{v} \left(v^2/2 + w \right)}$

Interesting to note:

→ unit mass of fluid carries energy $v^2/2 + w$ (<u>not</u> $v^2/2 + \epsilon$) in motion

→ from
$$w = \epsilon + pV$$
$$= \epsilon + \frac{p}{\rho}$$

Then, for energy flux thru surface:

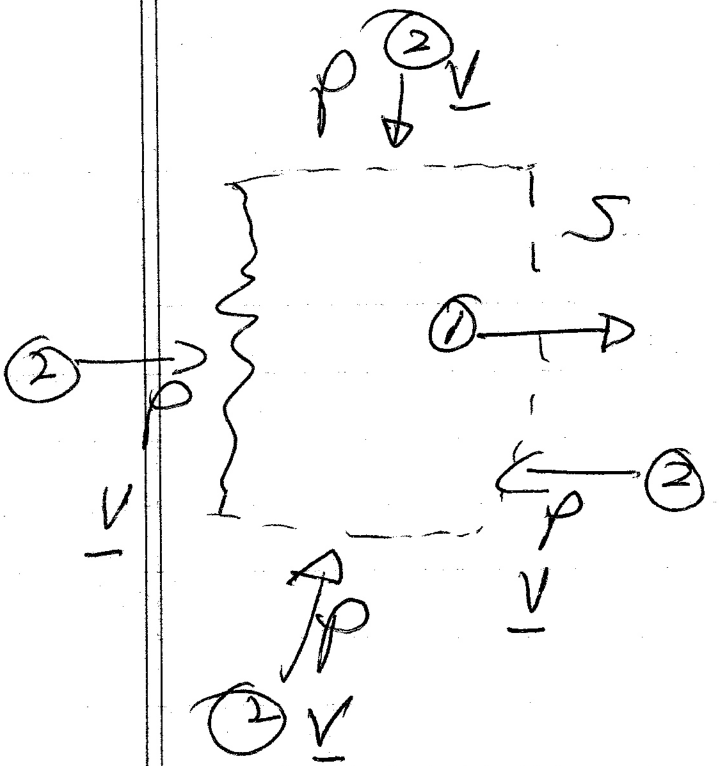
$$\int d\underline{s} \cdot \rho \underline{v} \left(\frac{v^2}{2} + w \right) = \int d\underline{s} \cdot \rho \underline{v} \left(\frac{v^2}{2} + \epsilon + \frac{p}{\rho} \right)$$

①
$$= \int d\underline{s} \cdot \rho \underline{v} \left(\frac{v^2}{2} + \epsilon \right)$$

→ naive expectation (kinetic + internal energy transported)

②
$$+ \int d\underline{s} \cdot \underline{v}\, p$$

→ work done <u>by</u> pressure forces on fluid <u>within</u> surface.



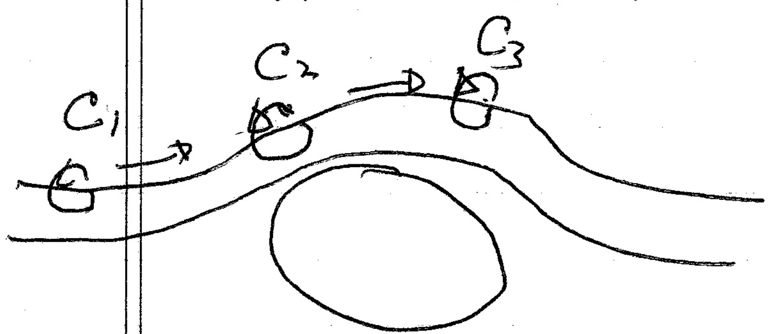
① i.e. transport thru surface

② i.e. work done by pressure forces on fluid within S.

~~(ii) Potential Flow in Ideal Incompressible Fluids~~

i.) Potential Flow

- Consider fluid stream lines:



if $\underline{\omega} = 0$ at any point along streamline, then Kelvin thm $\Rightarrow$ $\underline{\omega} = 0$ everywhere on streamline.

Easily seen by considering circulation around infinitesimal loop "pulled" along streamline. Thus, if

$$\oint_{C_1} \underline{v} \cdot d\underline{\ell} = \int_{A_1} \underline{\omega} \cdot d\underline{s} = 0, \quad \text{then} \oint_{C_n} \underline{v} \cdot d\underline{\ell} = \int_{A_n} \underline{\omega} \cdot d\underline{s} = 0$$

for all C_n.

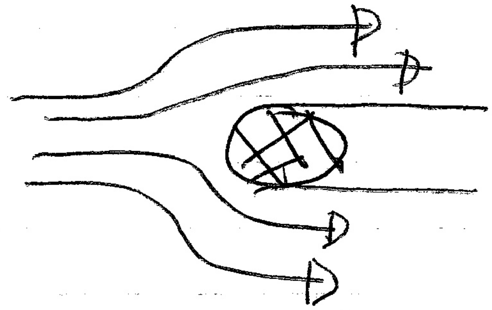
- flow with $\underline{\omega} = \underline{\nabla} \times \underline{v} = 0$ in all space is defined as:
  $\Rightarrow$ potential, irrotational flow

  $\Leftrightarrow$ $\underline{\omega} \neq 0$ rotational, vortical flow

- Important to note breakdown of Kelvin Thm

Applicability, namely to flows with separation

i.e. consider flow arnd. sphere

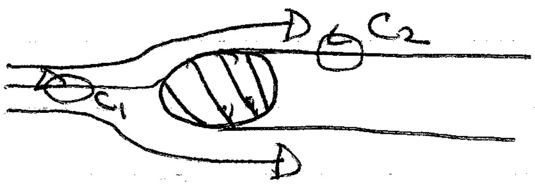


i.e. - streamlines separate from body

- surface of tangential discontinuity appears (velocity component)

⇒ - Kelvin Thm not applicable

i.e.



surface tangential discontinuity

- cannot infer $\oint_{C_2} \underline{v}\cdot d\underline{l}$ from $\oint_{C_1} \underline{v}\cdot d\underline{l}$ due to separation-induced tangential discontinuity.
- Also, viscosity important in (boundary layer) region of discontinuity. As viscous effects $\sim \nu k^2$, deviation from potential flow naturally most significant in small scale region of boundary layer!

Now, for isentropic fluids:

$W \equiv$ enthalpy

$$\partial \underline{v}/\partial t + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\underline{\nabla} W$$

For potential flow, $\underline{V} = \underline{\nabla}\phi \Rightarrow \underline{\omega} = 0$

stream function

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{V}\times\cancel{\underline{\omega}} + \underline{\nabla}(V^2/2)$$

$$= 0 + \underline{\nabla}(V^2/2)\,, \quad \text{for potential flow}$$

$\therefore$

$$\frac{\partial \underline{V}}{\partial t} + \underline{\nabla}\left(\frac{V^2}{2}\right) = -\underline{\nabla}W$$

$$\underline{V} = \underline{\nabla}\phi$$

$$\Rightarrow \underline{\nabla}\left(\frac{\partial\phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} + W\right) = 0$$

$\therefore$ have equation for dynamics of potential flow!

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} + W = f(t)}$$

$f(t)$ defined for each streamline

- for $\partial\phi/\partial t = 0$, recover Bernoulli's Law
- obvious that potential not uniquely defined, as $\underline{V} = \underline{\nabla}\phi$

Now, consider incompressible fluid potential flow, i.e.

- flows leaving density constant (no compression, expansion)

- $\nabla \cdot \underline{V} = 0 \iff \frac{d\rho}{dt} = 0$ (ρ constant)

$\Rightarrow$ if $\underline{V} = \nabla \phi \Rightarrow$

$$\boxed{\begin{gathered} \nabla^2 \phi = 0 \\ \frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} + \frac{P}{\rho} = f(t) \end{gathered}}$$

$\therefore$ for static flow, with gravity, $\Rightarrow$ Bernoulli Eqn.:

$$\boxed{V^2/2 + P/\rho + gz = \text{Const.}}$$

Criterion for "incompressibility":

- "incompressibility" is valid description for certain classes of flows, dependent on time scales, speeds, etc.

- for stationary flows $\quad \partial \underline{V}/\partial t = 0$,

$$\frac{P}{\rho} + \frac{V^2}{2} = \text{const.}$$

Now, for $\left\{\begin{array}{l}\text{adiabatic}\\ \text{isentropic}\end{array}\right\}$ fluid (T const) $\Rightarrow$

$$\Delta P = \left(\frac{\partial P}{\partial \rho}\right)_s \Delta \rho$$

but $\quad \frac{P}{\rho} + \frac{V^2}{2} = \text{const.}$

$$\Rightarrow \quad \Delta\left(\frac{V^2}{2}\right) = -\left(\frac{\partial P}{\partial \rho}\right)_s \frac{\Delta \rho}{\rho}$$

"Incompressibility" $\Rightarrow$ $\Delta\rho/\rho \ll 1$

$(\partial P/\partial \rho)_s = c_s^2$ (sound speed in fluid)

$\therefore \quad V^2/c_s^2 \ll 1 \Rightarrow$ flow incompressible.

Note: $\left\{\begin{array}{l}\text{Supersonic flows always compressible} \Rightarrow\\ \text{fluid dynamics couples to acoustic waves.}\end{array}\right.$

- for dynamic flows (more generally);

need compare terms in continuity equation

$$\frac{d\rho}{dt} = -\rho \, \underline{\nabla} \cdot \underline{V}$$

Now $\tau \to$ time scale for flow
$\ell \to$ spatial scale for flow

Then,
$$\frac{d\rho}{dt} \sim \frac{\Delta\rho}{\tau}$$

$$\rho\,\nabla\cdot\underline{V} \sim \rho\,\frac{\tilde{V}}{\ell}$$

To relate $\Delta\rho$ to $\tilde{V}$, consider Euler equation:

$$\frac{\partial \underline{V}}{\partial t} = -\frac{\nabla P}{\rho}$$

$$\Rightarrow \frac{\tilde{V}}{\tau} \sim \frac{c_s^2\,\Delta\rho}{\rho\ell} \qquad (\text{tacitly, } \tilde{V} \lesssim \ell/\tau)$$

$$\tilde{V} \sim \frac{\tau c_s^2\,\Delta\rho}{\rho\ell}$$

$$\therefore \quad \frac{d\rho}{dt} \sim \left(\frac{c_s^2\tau}{\ell\rho}\right)^{-1} \frac{\tilde{V}}{\tau}$$

$$\rho\,\nabla\cdot\underline{V} \sim \frac{\tilde{V}}{\ell}\,\rho$$

$$\nabla\cdot\underline{V} \cong 0 \quad \text{if} \quad \frac{d\rho}{dt} \ll \rho\,\nabla\cdot\underline{V}$$

$$\frac{\tilde{V}}{\ell}\rho \overset{?}{\gg} \frac{\ell\rho}{c_s^2\tau^2}\tilde{V} \Rightarrow c_s^2 \gg \frac{\ell^2}{\tau^2}$$

Thus, dynamics incompressible if $\begin{cases} c_s^2 \gg \ell^2/\tau^2 \\ \quad \gg \omega^2/k^2 \text{ (wave)} \end{cases}$

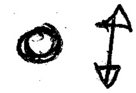
- note can synthesize static, dynamic conditions to obtain incompressibity criterion:

$$c_s^2 > \begin{cases} \tilde{V}^2 \\ \ell^2/\tau^2 \end{cases}$$

i.e. time slow compared to time to traverse 1 spatial scale at acoustic speeds.

Some further facts about potential flows (generally incompressible):

- for body (i.e. rigid sphere) immersed in fluid, if amplitude oscillation ≪ dimensions of body ⇒ motion describable by potential flow



i.e. $a \equiv$ amplitude motion
$u \equiv$ body velocity
$f \equiv$ frequency of oscillation
$\ell \equiv$ size of body

Simply compare $\frac{\partial \underline{V}}{\partial t}$ to $\underline{V}\cdot\underline{\nabla}\,\underline{V}$, noting

If $\partial \underline{V}/\partial t \gg \underline{V}\cdot\underline{\nabla}\underline{V} \Rightarrow \frac{\partial \underline{V}}{\partial t} \cong -\underline{\nabla} W$

so

$$\underline{\nabla}\times\underline{V} = 0 \Rightarrow \left\{ \begin{array}{l}\text{Potential} \\ \text{flow}\end{array}\right.$$

Now $\omega \sim u/a$

$\partial \underline{V}/\partial t = -i\omega \underline{V} \sim u^2/a$ ($\underline{V}\sim \underline{u}$ near body)

$\underline{V}\cdot\underline{\nabla}\underline{V} \sim u^2/\ell$ (ℓ sets smallest scale in problem)

$$\left|\frac{\partial \underline{V}}{\partial t}\right| \overset{?}{\gg} |\underline{V}\cdot\underline{\nabla}\underline{V}| \Rightarrow \frac{u^2}{a} \overset{?}{\gg} \frac{u^3}{\ell}$$

$$\Rightarrow \ell \gg a$$

Thus, fluid dynamics resulting from small oscillation of body describable by potential flow.

- In potential flow, streamlines must be open, not closed.

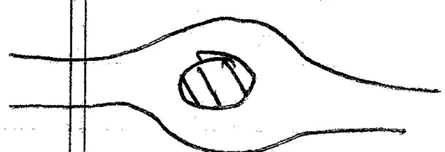
To see, consider circulation about closed contour

$$\oint_C \underline{V}\cdot d\underline{l} = \int_{S_{encl.}} d\underline{S}\cdot\underline{\omega} = 0$$

$\underline{\omega} = 0$ for potential flow

but, by definition, $\int_{\text{streamline}} \underline{V}\cdot d\underline{l} \neq 0$. $\Rightarrow$ streamlines must be open!

i.e.



sphere in $\underline{V} = V_0\hat{z}$ flow is typical potential flow problem (describes flow at distance from sphere).

- For incompressible flow, (not potential)

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\nabla\underline{V} - \underline{\omega}\,\cancel{\nabla\cdot\underline{V}}$$

In 2D, $\underline{\omega}\cdot\nabla\,\underline{V} = 0$ i.e. $\left\{ \begin{array}{l} \underline{V} = (V_x(x,y),\ V_y(x,y)) \\ \underline{\omega} = \omega_z(x,y)\hat{z} \end{array} \right.$

Then, $\frac{d\omega}{dt} = 0$

Now, $\nabla\cdot\underline{V} = 0 \quad \Rightarrow \quad \begin{array}{l} V_x = \partial\psi/\partial y \\ V_y = -\partial\psi/\partial x \end{array}$

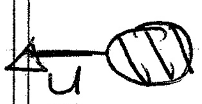
$$\underline{\omega} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z} = \hat{z}\left(-\nabla^2\psi\right)$$

$$\frac{d\underline{\omega}}{dt} = 0 \Rightarrow \begin{cases} +\dfrac{\partial}{\partial t}\nabla^2\psi + \underline{\nabla}\psi\times\hat{z}\cdot\nabla\nabla^2\psi = 0 \\ \text{2D incompressible fluid eqn.} \end{cases}$$

iv.) Problems in Potential Flow

a.) Incompressible Potential Flow Around Sphere

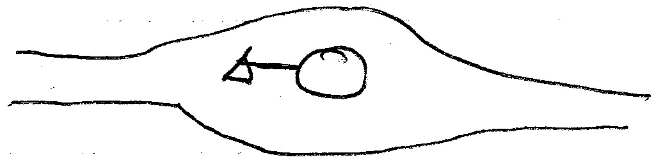
Consider rigid sphere in motion at $\underline{u}$ in infinite fluid



Flow Pattern ?

Now :

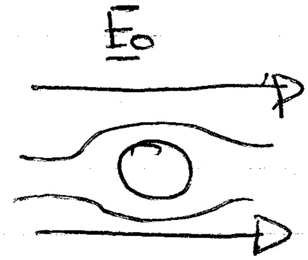
- intuitively, expect :



i.e. equivalent to $\begin{cases} \text{sphere at rest} \\ \underline{V}\big|_{\text{fluid}, \infty} = -\underline{u} \end{cases}$

Electrostatic analogy: Conducting sphere in uniform electric field

i.e.



$$\phi = -\underline{E}_0 \cdot \underline{r} + \phi_{sphere}$$

ϕ_{sphere} is dipole field.

Dipole moment determined by b.c.
i.e. ϕ = const = 0 on sphere surface

Now, for potential flow (incompressible):

$$\nabla^2 \phi = 0 \qquad \underline{V} = \nabla \phi$$

$$V_n = \underline{V} \cdot \hat{n} = \underline{u} \cdot \hat{n} \Big|_{surface}$$

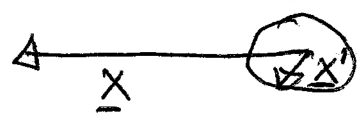
(i.e. normal velocity = sphere velocity on surface)

By analogy with electrostatics, can solve via:
- multipole expansion
- B.C.'s determine effective "charge" distribution

Recall e.s. $\Rightarrow$ $\nabla^2 \phi = -4\pi\rho$

$$\phi = \int d^3x' \, \frac{\rho(x')}{|\underline{x}-\underline{x}'|}$$

For $\underline{x}$ outside region ρ :



$$\phi(\underline{x}) = \int d^3x' \, \frac{\rho(x')}{|\underline{x}-\underline{x}'|}$$

$$= \int d^3x' \, \frac{\rho(x')}{|\underline{x}-\cancel{\underline{x}'}|} - \int d^3x' \, \underline{x}'\rho(x') \cdot \underline{\nabla}\left(\frac{1}{|\underline{x}|}\right) + \cdots$$

$$= \frac{Q}{|\underline{x}|} - \underline{d}\cdot\underline{\nabla}\left(1/|\underline{x}|\right) + \cdots$$

$\downarrow$ monopole $\quad$ $\downarrow$ dipole $\quad$ $\downarrow$ quadrupole

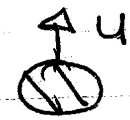
Thus, can write down general solution for potential flow streamlines around body as multipole expansion.

– $Q = 0$ (no sources, sinks)

$\therefore$ in general dipole dominates

– in 2D, same story with $\ln|x-x'| \to 1/|x-x'|$

Here: $\underline{v} = u\hat{z}$ (spherical symmetry)
(flow velocity) (body velocity)



$$\left.V_n\right|_R = \left.V_r\right|_R = u\hat{z}\cdot\hat{n} = u\cos\theta$$ boundary condition

Now, $\phi(\underline{x}) = \underline{A}\cdot\underline{\nabla}\left(1/|x|\right)$

$\underline{A} = A\hat{z}$ (dipole moment in $\hat{z}$ direction)

$\therefore$ $$\phi = -A\frac{\cos\theta}{r^2}$$

$$V_r = 2A\cos\theta/r^3$$

$$\Rightarrow \quad \frac{2A\cos\theta}{R^3} = u\cos\theta$$

$$\Rightarrow \quad A = \frac{R^3}{2}u$$

$\therefore$ $$\boxed{\begin{aligned}\phi &= -uR^3\cos\theta/2r^2 \\ \underline{v} &= \nabla\phi\end{aligned}}$$

determines general flow field

Note:

- can recover from $\phi = \sum_\ell \left(a_\ell r^\ell + b_\ell / r^{\ell+1} \right) P_\ell(\cos\theta)$ expansion and b.c.'s.

regularity at ∞

- if sphere in uniform field:

$$\phi = U_0 r\cos\theta + \phi_{sphere}$$

determine from $V_n = 0$

- to determine pressure distribution on sphere, recall:

$$\rho \frac{\partial \phi}{\partial t} + \frac{\rho V^2}{2} + p = p_0 \quad \Big\} \text{ incompressible Bernoulli Eqn.}$$

p_0 : ambient pressure at ∞

Thus, can immediately write:

$$p(\underline{x}) = p_0 - \frac{\rho}{2} \nabla\phi \cdot \nabla\phi - \rho \frac{\partial \phi}{\partial t}$$

- $\phi(\underline{x}) \equiv$ determined a la' above via $\nabla^2\phi = 0$ and b.c.'s.

As sphere in motion (but uniform) :

$$\frac{\partial \phi}{\partial t} = -\underline{u} \cdot \underline{\nabla} \phi + \cancel{\frac{\partial \phi}{\partial \underline{u}} \cdot \dot{\underline{u}}} \quad \dot{u} = 0$$

so

$$p(\underline{x}) = p_0 - \frac{\rho}{2} \underline{\nabla} \phi \cdot \underline{\nabla} \phi - \underline{u} \cdot \nabla \phi$$

Generally, leads to concept of stagnation point

i.e. for Bernoulli Eqn. for incompressible fluid :

$$\frac{P}{\rho} + \frac{V^2}{2} = \text{const.} = P_0$$

Now, consider fixed body in fluid with $\begin{cases} V_\infty = u_0 \\ P_\infty = P_0 \end{cases}$

As $V = 0$ on surface body :

$$P_{max} = P\Big|_{body} = P_0 + \frac{1}{2} \rho u^2$$

- stagnation point ($\underline{V} = 0$) on body is point of maximal pressure

- maximal pressure determined by $\begin{cases} P_0 \\ \text{speed} \end{cases}$

e.g. 

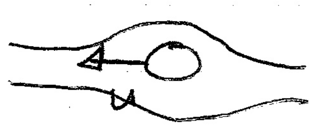
→ fish skeleton strongest on front face, weakest elsewhere

↔ front face is point of maximal pressure (head)

↔ eye lens adjusts to allow for speed-induced pressure changes.

b.) Drag Force and Induced Mass

→ Heuristics: Consider rigid body in water.



Slow body motion ⇒ potential flow around sphere
⇒ energy in fluid motion, too!

Thus, for F_{ext} to move body in fluid, need work against
- inertia of body (obvious)
- inertia of fluid, excited into potential flow

Thus, for body in water, need interpret Newton's 2nd Law as:

$$\underline{F}_{ext} = M_{eff} \frac{d\underline{u}}{dt}$$

$$M_{eff} = \underbrace{M}_{\text{mass of body}} + \overbrace{m_{induced}}^{\text{induced mass of fluid in potential flow around body}}$$

(mass of fluid flow which 'dresses' the body)

To calculate induced mass :

* – calculate energy in potential flow around rigid body in uniform motion in fluid

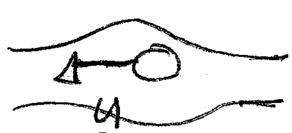
* – use $dE = d\underline{P} \cdot \underline{u}$ to determine momentum in fluid

as $\underline{P} = \underline{P}(\underline{u}) \Rightarrow P_i = m_{ik} u_k$

$\therefore$ m_{ik} is induced mass tensor !

→ Calculation : Consider rigid body moving in fluid

i.e.



Now, for flow field outside body, multipole expansion solution to $\nabla^2 \phi = 0$ yields

$$\phi = \cancel{\frac{c}{r}} + \underline{A}\cdot\underline{\nabla}\left(1/r\right) + \cdots$$

monopole (vanishes → no sources)

dipole (dominant multipole at large radius)

→ dipole moment : $A = c\,R_s^3\,\underline{u}$

($c = 1/2$, sphere)

$$\therefore\quad \phi = \underline{A}\cdot\underline{\nabla}\,(1/r)$$

$$= -\underline{A}\cdot\underline{r}/r^3 = -\underline{A}\cdot\hat{n}/r^2$$

$$\underline{V} = \underline{\nabla}\phi = \underline{A}\cdot\nabla\;\nabla(1/r)$$

$$= (\underline{A}\cdot\underline{\nabla})\,(-\underline{r}/r^3)$$

$$\underline{V} = \left(3(\underline{A}\cdot\hat{n})\,\hat{n} - \underline{A}\right)/r^3$$

Now, for energy, seek calculate fluid energy in volume V enclosed within radius R around body. Take $R^3 \gg V_0 \equiv$ volume of body.

Thus:

$$E = \frac{1}{2}\rho\int dV\,|\underline{\tilde{V}}|^2$$

$$= \frac{1}{2}\rho\int d^3x\left(\underline{u}^2 + |\tilde{V}|^2 - \underline{u}^2\right)$$

out $|\nabla\phi|^2 - u^2 = (\underline{V}+\underline{u})\cdot(\underline{V}-\underline{u})$

$= \nabla(\phi + \underline{u}\cdot\underline{r})\cdot(\underline{V}-\underline{u})$

$= \underline{\nabla}\cdot\left[(\phi+\underline{u}\cdot\underline{r})(\underline{V}-\underline{u})\right]$

as $\underline{V} = \underline{\nabla}\phi$ $\quad \underline{\nabla}\cdot\underline{V} = 0$

$\underline{u} = \text{const.}$ $\quad \underline{\nabla}\cdot\underline{u} = 0$

$$\therefore \quad E = \frac{1}{2}\rho\int d^3x\left[u^2 + \nabla\cdot\left[(\phi+\underline{u}\cdot\underline{r})(\underline{V}-\underline{u})\right]\right]$$

$$= \frac{1}{2}\rho u^2(V-V_0) + \frac{1}{2}\rho\int d\underline{s}\cdot\left[(\phi+\underline{u}\cdot\underline{r})(\underline{V}-\underline{u})\right]$$

→ Volume object/body

Volume space

$V = \frac{4\pi}{3}R^3$

$(\underline{V}-\underline{u})\cdot d\underline{s} = 0$ on R_0 surface

Now, $d\underline{s} = \hat{n}\,R^2 d\Omega$, on outer surface

$$E = \frac{1}{2}\rho u^2(V-V_0)$$

$$+ \frac{1}{2}\rho\int R^2 d\Omega\left[(\hat{n}\cdot\underline{V} - \hat{n}\cdot\underline{u})(\phi + \underline{u}\cdot\underline{R})\right]$$

$$\tilde{E} = \frac{1}{2}\rho u^2 (V - V_0)$$
$$+ \frac{1}{2}\rho \int R^3 d\Omega \left[\left(2\frac{(\underline{A}\cdot\vec{n})}{R^3} - \underline{u}\cdot\vec{n} \right) \left(\frac{-\underline{A}\cdot\vec{n}}{R^2} + R\,\underline{u}\cdot\hat{n} \right) \right]$$

$$= \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int R^2 d\Omega \left[\cancel{-2\frac{(\underline{A}\cdot\vec{n})^2}{R^5}} \right.$$ vanishes for large R

$$\left. + \frac{(\underline{u}\cdot\vec{n})(\underline{A}\cdot\vec{n})}{R^2} + \frac{2(\underline{A}\cdot\vec{n})(\underline{u}\cdot\vec{n})}{R^2} - R\,(\underline{u}\cdot\vec{n})^2 \right]$$

$$= \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int \cancel{R^3} d\Omega \left[\frac{3(\underline{A}\cdot\vec{n})(\underline{u}\cdot\vec{n})}{\cancel{R^2}} - R^3(\underline{u}\cdot\vec{n})^2 \right]$$

Thus finally,

$$E = \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int d\Omega \left[3(\underline{A}\cdot\underline{\vec{n}})(\underline{u}\cdot\underline{\vec{n}}) - R^3 (\underline{u}\cdot\vec{n})^2 \right]$$

$$d\Omega = d\theta \sin\theta \, d\phi$$

if $\int d\Omega \; (\;) = \langle (\;) \rangle$

$$\Rightarrow \langle (\underline{A}\cdot\vec{n})(\underline{B}\cdot\vec{n}) \rangle = \frac{1}{2}\delta_{ij} A_i B_j = \frac{1}{3}\underline{A}\cdot\underline{B}$$

$$E = \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho\left[4\pi \underline{A}\cdot\underline{u} - \frac{4\pi}{3}R^3 u^2\right]$$

$$= \frac{1}{2}\rho\left[4\pi\, \underline{A}\cdot\underline{u} - u^2 V_0\right]$$

Thus finally, $\boxed{E = \frac{\rho}{2}\left[4\pi\, \underline{A}\cdot\underline{u} - u^2 V_0\right]}$ energy in potential flow and body

Now, $\underline{A} = \underline{A}(u) \Rightarrow E = \frac{1}{2} m_{ik} u_i u_k$

defines induced mass tensor

$$dE = \underline{u}\cdot d\underline{P}$$

$$\Rightarrow \boxed{\underline{P} = \rho_{H_2O}\left[4\pi \underline{A} - V_0 \underline{u}\right]}$$

momentum in potential flow

Now, consider external force acting system, where system = body + fluid (in Pot. flow)

i.e. $\underline{f}_{ext} = \frac{d\underline{P}_{fluid}}{dt} + M_{body}\frac{d\underline{u}}{dt}$

$\Rightarrow \quad f_i = (M\delta_{ik} + m_{ik})\frac{du_k}{dt}$

∴ → effective mass of "system" is sum of
- body mass
- induced mass of fluid in potential flow around body

→ note induced mass is determined purely by body shape (i.e. via volume and dipole moment)

i.e. for sphere $\quad \underline{A} = \frac{R_0^3}{2}\underline{u}$

$$\underline{P} = \rho\left[4\pi\frac{R_0^3}{2}\underline{u} - \frac{4\pi}{3}R_0^3\underline{u}\right]$$

$$= \rho\,\frac{2}{3}\pi R_0^3\,\underline{u}$$

$m_{induced} = \rho\,\frac{2}{3}\pi R_0^3$

In general $M_{induced} \sim \# \rho R_0^3$

$$\sim \# \underbrace{\rho V_0}_{\text{displaced mass fluid}}$$

→ numerical factor, shape dependent

→ Example of "renormalization" in classical physics "dressing field" in continuum i.e. { renorm. polaron, debye shield, etc.

i.e. in quantum electrodynamics → electron polarizes vacuum

$$\to m_e = m_e^{bare} + m_e^{V.P.}$$
$$(E = mc^2)$$

in classical potential flow → moving a sphere in H_2O requires that some energy go into surrounding media (the water!)

(skip)

→ Enhanced inertia due induced mass may, alternatively, be viewed as drag force on body (careful of phases!)

mom. transmitted to fluid

i.e. $$f_{ext} = \frac{dP_{fluid}}{dt} + M\frac{d\underline{v}}{dt}$$

drag!

$$\therefore \quad M \frac{d\underline{u}}{dt} = \underline{f}_{ext} - \frac{d\underline{P}_{fluid}}{dt}$$

$$= \underline{f}_{ext} + \underline{f}_{drag,\, lift} \qquad \underline{f}_{drag} \sim \dot{u}$$

$$\underline{f}_{drag} = -\frac{d\underline{P}_{fluid}}{dt}, \text{ along direction motion.}$$

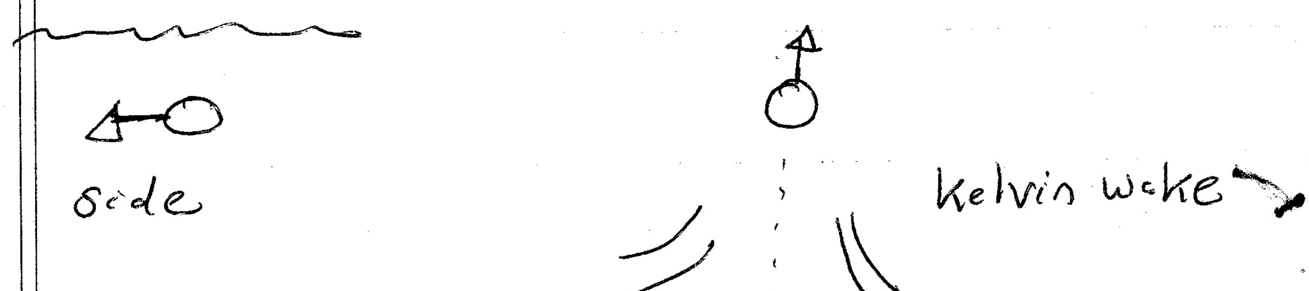
$$\underline{f}_{lift} = -\frac{d\underline{P}_{fluid}}{dt}, \ \perp \text{ direction of motion.}$$

Note: → if body in uniform motion in ideal (fantasy) fluid $\underline{f}_{drag} = \underline{f}_{lift} = 0$ } D'Alembert's Paradox

→ need no external force to maintain uniform motion

as
- no dissipation (ideal fluid)
- no loss of energy to ∞ $(v \sim 1/R^3)$

→ but if body near surface



body will radiate surface waves to ∞ (wake) ⇒ wave drag induced energy loss!

example: Obtain:

a) - eqn. of motion for sphere oscillating in fluid

b) - sphere in oscillating fluid

a) for sphere $\underline{A} = \frac{1}{2} R^3 \underline{u}$

for oscillating sphere

$$f_{ext} = m a_{sphere} + \underbrace{(m\dot{\underline{v}})_{induced}}_{\text{acceleration of dressing}}$$

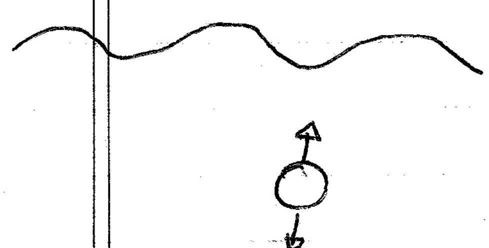
$$m\dot{\underline{v}} = M_{ind} \dot{\underline{u}}$$

$$M_{ind} = \underbrace{\frac{2}{3} \pi R^3 \rho_{H_2O}}_{\text{virtual mass}}$$

$$\therefore \quad f_{ext} = \frac{4\pi R^3}{3} \left(\rho_{sph} + \frac{\rho_{H_2O}}{2} \right) \frac{du}{dt}$$

→ Related Problem:

- consider body in fluid, which is set in motion by external agent



Relate $\underline{u}$ body to $\underline{V}$ fluid !?

- Now $\underline{V} \equiv$ velocity of unperturbed flow

$$\frac{\|\underline{\nabla}\,\underline{V}\|\, R_0}{\|\underline{V}\|} \ll 1 \quad \Rightarrow \underline{V} \sim \text{const over scale of body}$$

(potential flow valid)

So if body $\underline{\underline{\text{fully}}}$ carried along by fluid ($\underline{V}=\underline{u}$), then force on it would equal force on volume of displaced fluid

i.e. $$\frac{d}{dt}(M\underline{u}) = \rho V_0 \frac{d\underline{V}}{dt}$$

but body moves relative to fluid, so that fluid aquires momentum

→ drag duo relative motion

i.e. $$d\underline{P}_{fluid}/dt = -\underline{\underline{m}}\cdot\frac{d}{dt}[\underline{u}-\underline{V}]$$

∴ so really,

$$\frac{d}{dt}(M\underline{u}) = \rho V_0 \frac{d\underline{v}}{dt} - \underline{\underline{m}} \cdot \frac{d}{dt}(\underline{u}-\underline{v})$$

$$\frac{d}{dt}(Mu_i) = \rho V_0 \frac{dv_i}{dt} - m_{ik}\frac{d}{dt}(u_k - v_k)$$

$\Rightarrow$

$$Mu_i = \rho V_0 v_i - m_{ik}(u_k - v_k)$$

∴

$$(M\delta_{ik} + m_{ik})u_k = (\rho V_0 \delta_{ik} + m_{ik})v_k$$

$$\boxed{u_k = \left(\frac{\rho V_0 \delta_{ik} + m_{ik}}{M\delta_{ik} + m_{ik}}\right) v_k}$$

Note: $\rho V_0 < M$ (body heavier than displaced fluid) $\rightarrow$ body lags

$\rho V_0 > M$ $\rightarrow$ body leads

$\rho V_0 = M$ $\qquad u_k = v_k$.

Thus

$$M \frac{d\underline{u}}{dt} = \rho_f V \frac{d\underline{v}}{dt} - \underline{\underline{m}} \cdot \frac{d}{dt}\left[\underline{u} - \underline{v}\right]$$

$$(M \delta_{ij} + m_{ij}) \frac{du_j}{dt} = M_f \delta_{ij} + m_{ij} \frac{dV_j}{dt}$$

$$\therefore \quad u_j = \left[(M_f \delta_{ij} + m_{ij}) / (M \delta_{ij} + m_{ij}) \right] V_j$$

$$M_f = \rho_f V_0$$
$$M = \rho V_0$$

$\Rightarrow$ $\underline{u} = \underline{v}$ if $\rho_f = \rho$

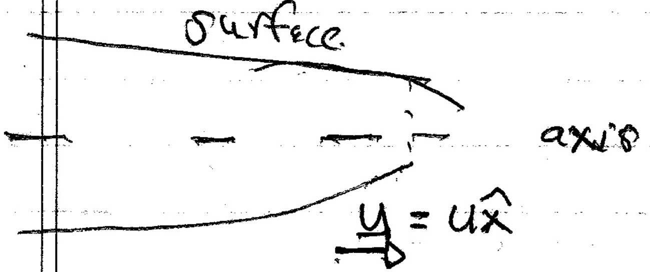
$u < v$ if $\rho_f < \rho$ $\rightarrow$ heavy object lags

$\rho_f \equiv$ fluid density
$\rho \equiv$ body density

$u > v$ if $\rho_f > \rho$ $\rightarrow$ light object leads.

c.) Potential Flow – General Slender Body

– Till now, have considered simple body potential flows, i.e. sphere, cylinder,
Here consider general body from surface of revolution



i.e. – general, axially symmetric slender body

– slender $\Rightarrow$ $W/L \ll 1$

Now, observe analogy with electrostatics again,

i.e. e.s. $\Rightarrow$ $\phi(x) = \int d^3x' \rho(x') / |\underline{x} - \underline{x}'|$

potential flow $(A \sim uV)$

$$\phi(x) = \frac{1}{4\pi} \int d^3x' \left(\dot{\rho}(x') / \rho_0 \right) / |\underline{x} - \underline{x}'|$$

$\frac{\dot{\rho}(x')}{\rho_0} \equiv$ normalized density of fluid flowing accross cross-section of body

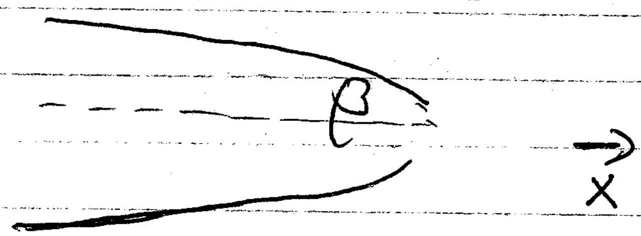
$\rightarrow$ yields $A \sim V_0 u$ etc.

$$\therefore \quad \phi(x) = \frac{1}{4\pi |x|^2} \int d^3x' \frac{\dot{\rho}(x')}{\rho_0} x' \quad + \text{ h.o.t.}$$

$\downarrow$

dipole term dominates

now, – body slender $\rightarrow \frac{W}{L} < 1 \Rightarrow \beta \ll 1$



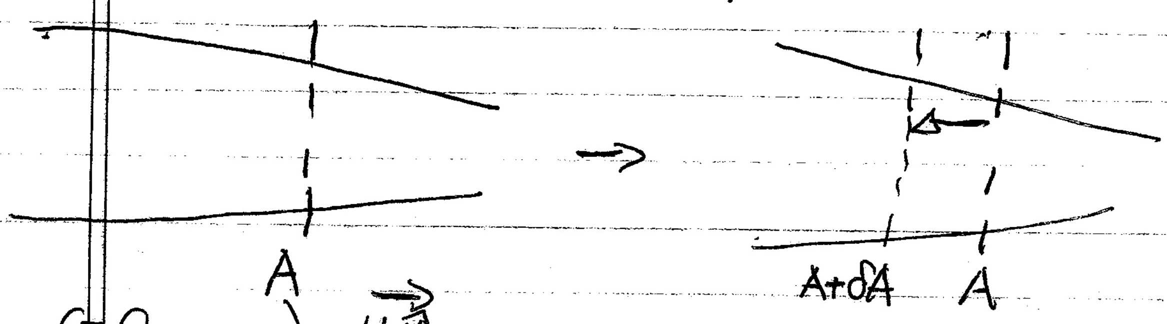
– $\nabla \cdot \underline{V} = 0$ and axial symmetry $\Rightarrow$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r V_r \right) = 0$$

$$\therefore \frac{V_r}{V_x} \sim \frac{\Delta r}{\Delta x} \sim \beta \sim \frac{W}{L} \ll 1$$

$\Rightarrow$ need only consider $\hat{x}$ fluid motion

$\therefore$ to compute dipole moment, need $\rho(x')/\rho_0$ for fluid flow accross body



(≡ faces area) $u\vec{\hat{x}}$

$$\text{Net } \frac{\dot{\rho}}{\rho_0} = u\left[A + \delta A - A \right] = u \frac{\partial A}{\partial x} dx$$

$$\Rightarrow \dot{\rho}(x')/\rho_0 = u \frac{\partial A}{\partial x}$$

$$\therefore \quad \phi(x) = \frac{1}{4\pi |x|^2} \int dx' x' u \frac{\partial A(x')}{\partial x'}$$

$$= \frac{-u}{4\pi |x|^2} \int dx' A(x')$$

$$= \frac{-u}{4\pi |x|^2} V$$

$V \equiv$ volume of body $= \int dx' A(x')$

$\Rightarrow$ yields intuitive result:

$$\phi(\underline{x}) = \underbrace{-u V_{body}}_{\text{effective dipole moment for slender body.}} / 4\pi r^2$$