4.) Hamilton - Jacobi Theory

Physics 200 A — Mechanics

I.) Integrability and Hamilton-Jacobi Theory; Principle of Maupertuis

a) Review

→ recall two perspectives on: Action and Principle of Least Action

① $S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$ $\qquad q(t_{1,2}) = q_{1,2}$

$\delta S = 0 \Rightarrow$ Lagrange Eqns.

⇒ "S as functional" ⟷ {fixed end-pts.

② $S = S(q, t)$

⇒ "S as function" ⟷ {variable upper end-point

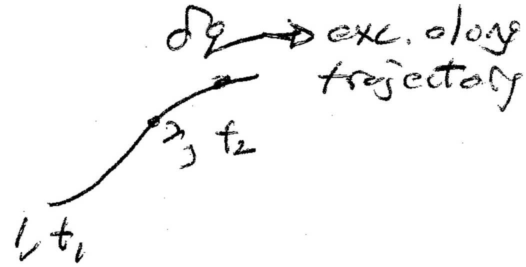
$S(q, t)$?

$$dS = \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt$$

so seek $\partial S/\partial q$, $\partial S/\partial t$ for basic parametrization of functional form.

Now, $\dfrac{\partial S}{\partial q} = \dfrac{\partial L}{\partial \dot{q}} = p$

$$\frac{\partial S}{\partial t} = -H$$



To see:

1) $$\delta S = \int_{t_1}^{t} dt \left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q} \right)$$

$$= \frac{\partial L}{\partial \dot{q}}\,\delta q \Big|_{t_1}^{t} + \cancelto{0}{\int_{t_1}^{t} dt \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right\} \delta q}$$

$$= \frac{\partial L}{\partial \dot{q}}\,\delta q = p\,\delta q$$ as $q(t)$ satisfies L.E.M.

as $\delta q(t_1) = 0$ (fixed lower end-point)

2) $S = S(q, t)$

$$\frac{dS}{dt} = \frac{\partial S}{\partial q}\,\dot{q} + \frac{\partial S}{\partial t}$$

but $dS/dt = L$

$\partial S/\partial q = p$

$$\equiv \quad L = p\dot{q} + \frac{\partial S}{\partial t}$$

$$\Rightarrow \quad \partial S/\partial t = -H \qquad\qquad H = p\dot{q} - L$$

Thus, have:

$$dS = \sum_i p_i dq_i - H dt$$

$$\Rightarrow \quad dS = dS_0 - H dt \quad , \text{etc.}$$

and, can proceed to develop: $\left\{ \begin{array}{l} \text{Abbreviated Action} \\ \text{Principle of Maupertuis} \\ \text{etc.} \end{array} \right.$

b.) Hamilton - Jacobi Theory $\rightarrow$ $\left\{ \begin{array}{l} \text{- insights into integrability of motion esp. in various geometries} \\ \text{- relation to QM.} \end{array} \right.$

$\rightarrow$ Recall $H = H(q, p, t)$

$$\text{where} \quad \left. \begin{array}{l} \dot{p} = -\partial H/\partial q \\ \dot{q} = \partial H/\partial p \end{array} \right\} \quad \text{Hamilton's Eqns.}$$

but have shown:

$$H = H(q, \partial S/\partial q, t)$$

so $$\frac{\partial S}{\partial t} = -H(p, q, t)$$

$$= -H\left(\frac{\partial S}{\partial q}, q, t\right)$$

contains full info. about dynamics $\Leftrightarrow$ i.e. all info. in Hamilton's Eqns.

Thus:

$$\boxed{\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q, t\right) = 0}$$

Hamilton-Jacobi Eqn.

If $\partial L/\partial t = 0$, so H conservative

$$H = H(p, q) = H\left(\frac{\partial S}{\partial q}, p\right) = E$$

$$\Rightarrow \boxed{H\left(\frac{\partial S}{\partial q}, p\right) = E}$$

(Time-Independent) Hamilton-Jacobi Equation for Conservative System

→ Comments on H.-J. equation: i.e. Why?

i.) single, first-order nonlinear pde has full content of dynamical system
$\Rightarrow$

ii.) solvability H-J eqn. $\Leftrightarrow$ integrability of dynamical system — geometrical insight

iii.) focus on techniques to obtain $S(q, t)$
$\Leftrightarrow$ equivalent to solving Hamilton's eqns.

but

techniques reveal structure of problem and properties of system rendering it amenable to integration.

iv.) Also, H-J. equation is eikonal equation for Schroedinger Eqn.

i.e. S.E.: $+i\hbar \dfrac{\partial \psi}{\partial t} = H\,\psi$

$$= -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

for $\hbar \to 0$ (semi-classical limit)

$$\psi = \psi_0 \, e^{i\phi(\underline{x}, t)/\hbar}$$

phase function $\Leftrightarrow$ rapid variation ($\hbar \to 0$ approx.)

$$\therefore \quad -\cancel{i\hbar}\frac{i}{\cancel{\hbar}}\frac{\partial\phi}{\partial t} = \frac{\cancel{\hbar^2}}{2m}\frac{(+1)}{\cancel{\hbar^2}}(\nabla\phi)^2 + V$$

eikonal eqn.

$$\therefore \quad \boxed{\frac{\partial\phi}{\partial t} = \frac{(\nabla\phi)^2}{2m} + V = H(\nabla\phi, \underline{x}, t)}$$

if take $+\cancel{S}(x,t) \equiv \phi(\underline{x},t)$, by classical correspondence then eikonal equation becomes H.-J. equation, i.e.

$$\frac{\partial S}{\partial t} = -H\left(\frac{\partial S}{\partial q}, q, t\right), \text{ etc.}$$

$\therefore \quad \psi(\underline{x},t) = \psi_0 e^{iS(x,t)/\hbar}$ is eikonal approximation to wave function.
S as eikonal phase.

→ Solving the H-J equation.

Consider conservative H-J equation, i.e. :

$$E = H(\partial S/\partial \underline{q}, \underline{q}), \text{ and } \quad q = (q_1, q_2, \dots)$$

i) For trivial example of 1D oscillator:

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}$$

H-J equation ⇒

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2} = E$$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = E - \frac{kq^2}{2}$$

$$\therefore \boxed{S = \sqrt{2m}\int dq\,\sqrt{E - kq^2/2} = S(q)}$$

but also $\partial S/\partial q = p = m\frac{dq}{dt}$

$$\therefore \frac{dq}{dt} = \sqrt{2m}\,(E - kq^2/2)^{1/2}$$

$$\Rightarrow \boxed{\int dt = \int dq \Big/ \sqrt{2m}\,(E - kq^2/2)^{1/2}}$$

i.e. formal soln.

clearly obtaining S is equiv. to solution.

ic.) Non-trivial solution ⟷ Separating Variables in H-J equation

> Pose Question: For what $V(r, \theta, \phi)$ is motion integrable, in spherical geometry?

Integrability $\Leftrightarrow$ H.-J. eqn. separable!
(ala' separation of variables in pdes)

Then,

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + U(r, \theta, \phi)$$

H.-J. eqn:

$$E = \frac{1}{2m}\left(\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \left(\frac{\partial S}{\partial \phi}\right)^2 \frac{1}{r^2 \sin^2\theta}\right)$$

$$+ U(r, \theta, \phi)$$

To separate: $\rightarrow S = S_1(r) + S_2(\theta) + S_3(\phi)$

$\rightarrow$ structure U must match factors

$\therefore$

$$\left. U = a(r) + \frac{b(\theta)}{r^2} + \frac{c(\phi)}{r^2 \sin^2\theta} \right\}$$ integrable/separable form.

Now, separating the H.-J. equation;

→ first, re-write:

$$E = \left(\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + a(r)\right) + \frac{1}{r^2}\left(\frac{1}{2m}\left(\frac{\partial S}{\partial \theta}\right)^2 + b(\theta)\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\left(\frac{1}{2m}\left(\frac{\partial S}{\partial \phi}\right)^2 + c(\phi)\right)$$

Now, observe if can decompose as:

$$S = S_1(r) + S_2(\theta) + S_3(\phi)$$

then:

$$E = \left\{\frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + a(r)\right\} + \frac{1}{r^2}\left\{\left\{\frac{1}{2m}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + b(\theta)\right\}\right.$$

$$\left. + \frac{1}{\sin^2\theta}\left\{\frac{1}{2m}\left(\frac{\partial S_3}{\partial \phi}\right)^2 + c(\phi)\right\}\right\}$$

so can write:

$$E = f_1(r) + \frac{1}{r^2}\left\{f_2(\theta) + \frac{1}{\sin^2\theta} f_3(\phi)\right\}$$

i.e. $\nabla^2\psi + \frac{\omega^2}{c^2}\psi = 0 \Rightarrow \underbrace{\frac{X''}{X}}_{-k_x^2} + \underbrace{\frac{Y''}{Y}}_{-k_y^2} + \underbrace{\frac{Z''}{Z}}_{-k_z^2} = -\frac{\omega^2}{c^2}$

so, if:

$f_3(\phi) = C_\phi \rightarrow$ a COM ($\Rightarrow p_\phi$)

$f_2(\theta) + \frac{C_\phi}{\sin^2\theta} = C_\theta \rightarrow$ a COM ($\Rightarrow p_\theta$)

$f_1(r) + C_\theta / r^2 = E \rightarrow$ a COM

can $\begin{cases} \text{1) solve azimuthal, polar and radial Eqn. motion} \\ \text{2) separate and solve H-J equation} \end{cases}$

Key Point:

$\rightarrow$ in separation of H-J equation, separation constants (C_ϕ, C_θ, E)
- related COM's (p_ϕ, L^2, E)
- related <u>symmetry</u>: $\begin{cases} \text{azimuthal} \\ \text{rotational} \end{cases}$

$\rightarrow$ "separation of variables solution" $\Leftrightarrow$ ability to define/identify COM for each degree of freedom of motion.

Proceeding:

$$f_3(\phi) = C_\phi$$

$$\Rightarrow \quad 1/2m \left(\frac{\partial S_3}{\partial \phi}\right)^2 + c(\phi) = C_\phi$$

Take $c(\phi) = 0$ i.e. no azimuthal symmetry-breaking in potential

so

$$1/2m \left(\frac{\partial S_3}{\partial \phi}\right)^2 = C_\phi$$

clearly $\dfrac{\partial S_3}{\partial \phi} = \text{const.} = \underbrace{p_\phi}_{\text{azimuthal momentum}} \Rightarrow S_3 = p_\phi \phi + c_3$

$$C_\phi = \frac{p_\phi^2}{2m}$$

Then, H-J equation becomes (upon absorbing $p_\phi^2/2m\sin^2\theta$ into S_2 piece)

$$E = \left\{\frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + a(r)\right\} + \frac{1}{r^2}\left\{\frac{1}{2m}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + b(\theta) + \frac{p_\phi^2}{2m\sin^2\theta}\right\}$$

observe: $$\frac{1}{2m}\left(\frac{\partial S_2}{\partial\theta}\right)^2 + b(\theta) + \frac{p_\phi^2}{2m\sin^2\theta}$$

$$= f_2(\theta) + \cancel{\frac{f_3(\phi)}{\sin^2\theta}} \to p_\phi^2/2m$$

$$\equiv f_2'(\theta)$$

For f_2' piece, separation $\Rightarrow$

$$\frac{1}{2m}\left(\frac{\partial S_2}{\partial\theta}\right)^2 + b(\theta) + \frac{p_\phi^2}{2m\sin^2\theta} = \text{const. of separation for } S_2$$

$$= L^2$$

clearly, COM here is angular momentum related

$\Rightarrow$

$$\frac{\partial S_2}{\partial\theta} = \left(L^2 - b(\theta) - \frac{p_\phi^2}{2m\sin^2\theta}\right)^{1/2}\sqrt{2m}$$

$$\therefore\ S_2 = \int d\theta\,\sqrt{2m}\left(L^2 - b(\theta) - \frac{p_\phi^2}{2m\sin^2\theta}\right)^{1/2} + C_2$$

observe: $\to \theta = \pi/2$, reality $S \Rightarrow p_\phi^2 \leq L^2$

$\to L^2$ is not explicitly angular momentum unless $b(\theta) = 0$

Then, absorbing L^2/r^2 into radial piece $f_1(r)$:

$$E = \frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + a(r) + \underbrace{L^2/2mr^2}_{\text{from } f_2'/r^2} \qquad (L^2 \to L^2/2m)$$

E ↓ final, universal C.O.M

$\Rightarrow$

$$S_1 = \int dr\, \sqrt{2m}\left(E - a(r) - L^2/2mr^2\right)^{1/2} + c_1$$

$\Rightarrow$

$$S = S_1(r) + S_2(\theta) + S_3(\phi)$$

$$= \int dr \left[\sqrt{2m}\left(\underset{\text{COM}}{E} - a(r) - \underset{\text{COM}}{L^2}/2mr^2\right)^{1/2}\right] + \int d\theta \left[\left(\underset{\text{COM}}{L^2} - b(\theta) - \frac{\underset{\text{COM}}{p_\phi^2}}{2m\sin^2\theta}\right)^{1/2}\sqrt{2m}\right] + \underset{\text{COM.}}{p_\phi}\,\phi + \text{const.}$$

$= S(r, \theta, \phi)$

is separation sol'n. of H.-J. equation for:

$U = a(r) + b(\theta)/r^2 + c(\phi)/r^2\sin^2\theta$

- separation constants, COM's are:

p_ϕ → separation const. for ϕ
→ related azimuthal momentum
(for $c(\phi) \neq 0$)

L^2 → separation const. for θ
→ related polar momentum
(for $b(\theta) \neq 0$)
→ $b = 0$, is angular momentum.

E → separation constant for r
→ energy.

→ can obtain explicit $q(t)$ from
$p = \partial S / \partial q$, etc.

c.)

H-J. Equation: Another Perspective

→ Recall, thrust of discussion is integrability
⟺

① H.-J. equation relevant, as separability of H-J. equation ⇒ integrability (converse not equivalent)

② more generally, integrability can render
1) all coordinates cyclic
2) all conjugate momenta constant

i.e. integrability if can find transformation such that:

$$\underbrace{p_i,\ q_i}_{\text{arbitrary g.c.'s}} \longrightarrow \overset{\text{gen. momentum}}{\alpha_i},\ \overset{\text{position}}{\beta_i} = \alpha_i t + t_0$$

such that:

$$\frac{d}{dt}\alpha_i = 0$$

Then, will show that H-J. equation is generating function of canonical transformation

$$p_i, q_i \longrightarrow \alpha_i, \beta_i$$

i.e. clearly, for conservative system, such a transformation must leave:

$$H(p_i, q_i) = H'(\alpha_i)$$

but : $H(p_i, q_i) = E = E(\alpha_i)$

$$\Rightarrow \boxed{H(p_i, q_i) = E(\alpha_i)}$$

$\rightarrow$ $\Big\{$ but this is just time-independent H-J equation

$\Rightarrow$

Technical Preliminaries :

a.) Poisson Brackets

b.) Canonical Transformations and Generating Fctns.

a.) Poisson Brackets

Recall :

- fundamental notion/concept/fact of Hamiltonian mechanics is
  - $\rightarrow$ incompressibility of phase space flow (Liouville's Thm.)

i.e.

$$\underline{V}_\Gamma = (\dot{q}_i, \dot{p}_i)$$

$$\underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = \frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$$

→ Abbreviated Action / Principle of Maupertuis

Now, $\delta S = 0$ (Principle Least Action) $\Rightarrow \left\{ \begin{array}{l} \text{"Path"} \\ \text{Position} \end{array} \right.$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Updownarrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ trajectory

Position : $\underline{q}(t)$

Path : $\underline{r}(\ell)$ → curve followed by particle (but not when particle at particular point) (i.e. geodesic)

$\partial_t L = 0 \Rightarrow H(p, q) = \underline{E}$

Now $\delta \int_{q_1, t_1}^{q_2, t_2} L = 0$

fixed endpoints ⇒ $\delta S = 0$

but if allow $\underline{\underline{t_2}}$ to vary:

$$\delta \int_{q_1, t_1}^{q_2, t} L = -H \delta t$$

(virtual paths $q_1 \to q_2$ but t variable)

i.e. particle passes thru q_2, but not necessarily at $\underline{\underline{t_2}}$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Downarrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ "path"

(i.e. $dS = \int p\,dq - \int H\,dt$)

or, for energy conserving virtual paths:

$$\delta S + E\,\delta t = 0$$

Now also: $S = \int \sum_i p_i dq_i - E(t - t_0)$

$$S_0 = \int \sum_i p_i dq_i \equiv \text{abbreviated action}$$

⇒ for paths:

$$\boxed{\delta S_0 = \delta \int \sum_i p_i dq_i = 0}$$

Principle of Maupertuis

→ abbreviated action has minimum with respect to all paths which conserve energy and pass thru final point at any t.

→ to use, need express momenta in terms q, dq, via

$$p_i = \frac{\partial}{\partial \dot{q}_i} L(q, \dot{q})$$

$$E(q, \dot{q}) = E$$

i.e.

$$L = 1/2 \sum_{i,k} a_{ik}(q)\, \dot{q}_i\, \dot{q}_k - U(q)$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_k a_{ik}(q)\, \dot{q}_k$$

so

$$E = 1/2 \sum_{i,k} a_{ik}(q)\, \dot{q}_i\, \dot{q}_k + U(q)$$

$$\Rightarrow \quad E - U = 1/2 \sum_{i,k} a_{ik}(q) \frac{dq_i\, dq_k}{(dt)^2}$$

$$\therefore \quad dt = \left(\sum a_{ik}\, dq_i\, dq_k \Big/ 2(E-U) \right)^{1/2}$$

Thus, can write:

$$dS_0 = \sum_i p_i\, dq_i$$

$$= \sum_k a_{ik}(q)\, \dot{q}_k\, dq_i = \sum_k a_{ik}(q) \frac{dq_k}{dt}\, dq_i$$

plugging in dt $\Rightarrow$

$$\Rightarrow \boxed{S_0 = \int \left[2(E-U) \sum_{ik} a_{ik}\, dq_i\, dq_k \right]^{1/2}}$$

→ Variational for Path

For single particle: $T = \frac{1}{2} m \underbrace{(d\ell}_{\text{path element}}/dt)^2$

$$\Rightarrow \boxed{\delta S_0 = \delta \int_{q_1}^{q_2} [2m(E-U)]\, d\ell = 0}$$

- Jacobi's Integral

- $U = 0 \Rightarrow \delta S_0 = \delta \int d\ell = 0$
  Path of Least Action is Geodesic!

N.B. Can get orbit from dt eqn.

Example: Differential Eqn. for Path?

$$\delta \int (E-U)^{1/2} d\ell$$

$$= -\left[\int \frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta \underline{r}}{2(E-U)^{1/2}}\, d\ell - (E-U)^{1/2}\, d\,\delta\ell \right]$$

but $\quad dl^2 = dr^2$

$$dl\, d\delta l = d\underline{r} \cdot d\, \delta\underline{r}$$

$$d\,\delta l = \frac{d\underline{r}}{dl} \cdot d\delta\underline{r}$$

$\Rightarrow$

$$\delta \int \left(\sqrt{E-U}\right) dl =$$

$$- \int \left\{ \frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta \underline{r}}{2\sqrt{E-U}}\, dl - \sqrt{(E-U)}\, \frac{d\underline{r}}{dl} \cdot d\delta \underline{r} \right\}$$

$\perp$ BP

$$0 = - \int \left\{ \frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta \underline{r}}{2\sqrt{E-U}}\, dl + \frac{d}{dl}\left[\sqrt{E-U}\, \frac{d\underline{r}}{dl}\right] \cdot \delta \underline{r}\, dl \right\}$$

$$\Rightarrow \boxed{2\,(E-U)^{1/2} \frac{d}{dl}\left[\sqrt{E-U}\, \frac{d\underline{r}}{dl}\right] = -\frac{\partial U}{\partial \underline{r}}}$$

$\rightarrow$ equation for path !.

N.B.: For eikonal theory:

$$\delta S_0 \Rightarrow \delta \Phi_0 = \delta \int \underline{k} \cdot d\underline{x}$$

eqn. for ray path. Need eliminate $\underline{k}$ in terms ω, $n(\underline{x})$, etc. to actually obtain equation for ray.

Summary - Variational Principles of Mechanics

i.) ('Standard') Principle Least Action

$$\delta \int_{q_1, t_1}^{q_2, t_2} dt\, L = 0$$

(fixed e.p.)

$\Rightarrow$ Lagrange Eqns.
Hamilton Eqns.
Liouville's Thm.

$\Leftrightarrow$ trajectory, phase space flow

ii.) $$\delta \int_{q_1, t_1}^{q, t} L\, dt = \delta S \qquad S(q, t) \quad \left\{ \begin{array}{l} \text{upper E.P.} \\ \text{Variable} \end{array} \right.$$

$$\Rightarrow \frac{\partial S}{\partial t} + H(\nabla_q S, q, t) = 0$$

Hamilton-Jacobi Theory

$\Leftrightarrow$ integrability, especially in different geometries.

iii.) $$\delta S_0 = \delta \int p\, dq = 0 \qquad \left\{ \begin{array}{l} \text{no time} \\ \text{specified} \end{array} \right.$$

$\Rightarrow$ path equation - curve of trajectory with no time specification

$\Leftrightarrow$ ray paths, etc., in optics.