Nonlinear Oscillators

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Nonlinear Oscillations

1D

P. Diamond

I.) Nonlinear Oscillations – An Introduction

a.) Overview / Roadmap

Can formulate sequence of problems:

1.) Simple harmonic oscillator – linear

→ external force < deterministic, random

$$\ddot{x} + \alpha \dot{x} + \omega_0^2 x = f_{ext}$$

α ← damping (friction const.)

ω_0^2 → k/m

<u>if</u> $f_{ext} = f e^{i\gamma t}$ ⇒
$$\begin{cases} x = A\, e^{i\gamma t} \\ A = f/m(\omega_0^2 - \gamma^2 + 2i\alpha\gamma) \end{cases}$$

2.) Variation: SHO (Linear), with <u>Parametric Variation</u>

<u>i.e.</u> $\omega_0^2 = \omega_0^2(t)$

$\omega_0^2(t)$:

- slow variation $\frac{1}{\omega_0^2}\frac{d}{dt}\omega_0^2 \ll \omega_0$ ⇒ adiabatic theory
- fast variation $\frac{1}{\omega_0^2}\frac{d}{dt}\omega_0^2 > \omega_0$ ⇒ parametric instability
  $\sim 2\omega_0$

For parametric instability, prototype system is:

$$\begin{cases} \ddot{x} + \omega^2(t)\, x = 0 \\ \omega^2(t) = \omega_0^2(1 + h\cos\gamma t) \\ \gamma \sim 2\omega_0 + \epsilon, \quad \epsilon \ll \omega_0 \end{cases}$$

$\omega^2 = \omega^2(t)$
$\Rightarrow \partial L/\partial t \neq 0$
$\Rightarrow$ energy not const., here

Mathieu's Eqn.

{ instability for $\gamma \cong 2\omega_0$ resonance

then true nonlinear problems.

3.) Nonlinear Oscillator - Conservative

i.e. $\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0$ — Duffing's Equation

ϵx^3: anharmonic / nonlinear } term

Here, $\rightarrow$ conservative system, with bounded phase space trajectories

i.e. $\frac{1}{2}\dot{x}^2 + F(x) = E$ — $\rightarrow$ quasi-periodic orbits bounded

$F(x) \rightarrow \omega_0^2 x^2/2 + \epsilon x^4/4$

$\rightarrow$ secularity in perturbation theory

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0$$

$$\bar{\omega}^2 = \frac{\omega_0^2\langle x^2\rangle + \epsilon\langle x^4\rangle}{\langle x^2\rangle}$$

$\leftarrow x^{(0)} \sim a \sin\omega_0 t$

$\epsilon x^3 \sim a^3 \sin^2\omega_0 t \sin\omega_0 t$

$\sim a^3 \sin\omega_0 t$ (on resonance)

$\Rightarrow$ secularity $\sim t\cos\omega_0 t$? ! ?

$\therefore$

$\Rightarrow$ nonlinear frequency shift ! $\leftrightarrow$ removes secularity !

$\rightarrow$ simplest example of Reductive P.T.

$$\Rightarrow \quad \omega = \omega_0^{\circ} + \omega^{(2)}$$

$\omega^{(2)}$: 2nd-order correction $\sim \frac{3\epsilon}{8\omega_0} a^2$ (amplitude dep.)

i.e. $\omega_0^2 x + \epsilon x^3$
$\rightarrow (\omega_0^2 + \epsilon \overline{x^2}) x$
$\sim (\omega_0^2 + (\#) \epsilon a^2) x$
$\mapsto$ effective ω-correction!

4.) Forced Nonlinear Oscillator

i.e. $\ddot{x} + \alpha \dot{x} + \omega_0^2 x + \epsilon x^3 = f_{ext}(t)$ — Forced Duffing's Equation

ϵx^3: NL; $f_{ext}(t)$: forcing

Here $\rightarrow$ role of nonlinearity / anharmonicity in resonance

$\rightarrow$ i.e. contrast: forced SHO $(\epsilon = 0)$

$f_{ext} = f e^{i\omega t}$; $x = B e^{i\omega t}$

$B = b e^{i\delta}$

dissipation resolves resonance

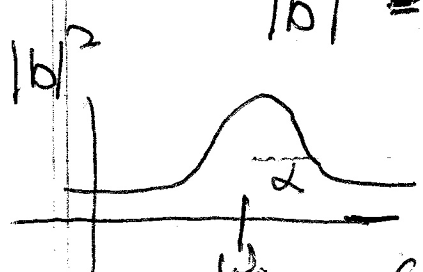
$$b = f/m \left[(\omega_0^2 - \omega^2)^2 + 4\alpha^2 \omega^2 \right]^{1/2}$$

$$\tan \delta = 2\alpha\omega / (\omega^2 - \omega_0^2)$$

so:

$$|b|^2 = f^2 / m^2 \left[(\omega_0^2 - \omega^2)^2 + 4\alpha^2 \omega^2 \right] \rightarrow \frac{f^2}{4m^2\omega_0^2} \left(\frac{1}{(\omega_0 - \omega)^2 + \alpha^2} \right)^{*}$$

(near resonance)



resonance $\rightarrow \omega_0$

linewidth $\rightarrow \alpha$

(symmetric)

but for NL forced SHO;

$\Rightarrow$ nonlinear frequency shift enters resonance!
here:

$$\omega-\omega_0 \to \omega-\omega_0 - K b^2$$

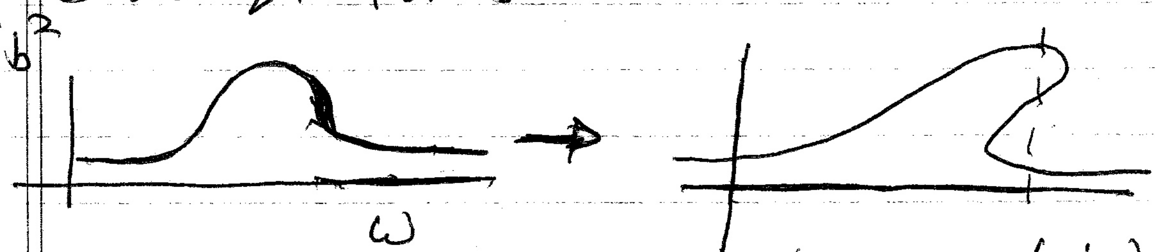
{amplitude-dependent shift

$\Rightarrow$

$$b^2 = \left(\frac{f^2}{4m^2\omega_0^2}\right) 1\Big/\left[(\omega-\omega_0-Kb^2)^2+\alpha^2\right]$$

$\to$ cubic eqn. for b^2

$\Rightarrow$



(symmetric) $\to$ (asymmetric)

S-curve, bifurcations, mode jumping, etc.

6.) Nonlinear Oscillator with ± Dissipation

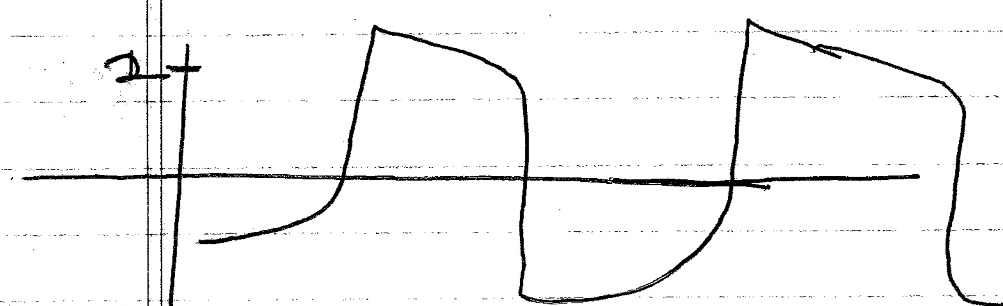
e.g. Van-der-Pol oscillator/Eqn.:

$$\ddot{x} + \alpha(x^2-1)\dot{x} + \omega_0^2 x = 0$$

key feature: $\underline{\text{nonlinear variable sign}}$ > $\underline{\text{dissipation}}$ (friction)

$$\Rightarrow \begin{cases} x^2 < 1 \rightarrow \alpha(x^2-1) < 0 \Rightarrow \begin{cases}\text{negative dissipation} \\ \text{instability}\end{cases} \\ \\ x^2 > 1 \rightarrow \alpha(x^2-1) > 0 \Rightarrow \begin{cases}\text{positive dissipation} \\ \rightarrow \text{stability / saturation}\end{cases} \end{cases}$$

$\Rightarrow$ bursty behavior, characteristic of limit cycle (at $x^2 = 1$)



Note:
In contrast to previous examples, system is fundamentally dissipative – nonlinearity in friction.

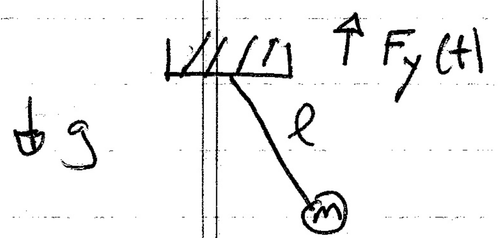
System/Eqn.	Nonlinearity	Externals
Mathieu (Parametric)	None	$\omega_0^2 = \omega_0^2(t)$ fast
Duffing (Freq. Shift)	Potential	None
Forced Duffing (Bi-stability/Bifurcation)	Potential	Forcing
Van-der-Pol (Limit Cycle)	Friction	None

Parametric Resonance and Instability

iv.) Parametric Instability

→ consider pendulum with support acted on by vertical force

$\downarrow + \Rightarrow$ down



so $g \to g - F_y(t)/m$

$$\therefore 0 = \ddot{\theta} + \frac{g}{\ell}\theta \;\rightarrow\; \ddot{\theta} + \left(\frac{g}{\ell} - \frac{a(t)}{\ell}\right)\theta = 0$$

→ let $a(t) = a_0 \cos(\alpha t)$

$$\Rightarrow \ddot{\theta} + \omega_0^2 \theta - \frac{a_0}{\ell}\cos(\alpha t)\,\theta = 0$$

$\therefore$ of Mathieu's equation genre, i.e.

$$\ddot{x} + \omega_0^2(1 + a\cos(\gamma t))\,x = 0$$

$$\Downarrow$$

$\omega^2 = \omega^2(t)$, hence parametric oscillator

Parametric oscillator $\leftrightarrow$ $\omega^2(t)$ periodic

→ Some observations:

ii) informal – consider what might happen?

for instability, observe can produce secularity if $\gamma \sim 2\omega_0$, via beat at fundamental

$$\underbrace{\ddot{x} + \omega_0^2 x}\, + a\cos(\gamma t)\,\omega_0^2 x = 0$$

$\cos(2\omega_0 t)$ $\sin(\omega_0 t)$

$\sim \sin(\omega_0 t)$

resonant drive of fundamental oscillator

$\Rightarrow$ secularity $\rightarrow$ instability (why?)

$\therefore$ Solution of oscillator at ω_0 beats with parameter oscillation $\Rightarrow$ secularity

$\therefore$ parametric resonance at/near $\gamma \sim 2\omega_0$ (twice fundamental)

Note: here $\omega^2 = \omega^2(t)$ $\Rightarrow$ $\partial L/\partial t \neq 0$ energy not conserved

$\Rightarrow$ work done on system (e.g. LGM oscillating pendulum support)

$\Rightarrow$ source of energy for instability

? What is relation of this to 3-mode parametric instability calculation (2004) ?

b) formal (Floquet theory) $\left\{ \begin{array}{l} \text{what} \\ \text{Mathematics} \\ \text{Predicts} \end{array} \right.$
(what type solution possible)

- $\omega(t)$ periodic, with period $T = 2\pi/\gamma$

$\therefore \left\{ \begin{array}{l} \omega(t+T) = \omega(t) \\ \text{eqn. invariant under } t \to t+T \end{array} \right.$

$\therefore$ if $x_1(t)$, $x_2(t)$ are 2 independent solutions of basic eqn.

$\Rightarrow$ $x_1(t)$, $x_2(t)$ must transform to linear combinations of themself upon $t \to t+T$ (linear eqn.)

and

can choose x_1, x_2 s/t

$$x_1(t+T) = \mu_1 x_1(t)$$
$$x_2(t+T) = \mu_2 x_2(t)$$

(here "can choose" means can diagonalize transformation matrix)

$\Rightarrow$ most general functions having this property are:

$$x_1(t) = \mu_1^{t/T}\, \Pi_1(t)$$

$$x_2(t) = \mu_2^{t/T}\, \Pi_2(t)$$

$\left\{ \begin{array}{l} \text{where:} \\ \Pi_{1,2}(t+T) = \Pi(t) \end{array} \right.$

– second, observe since linear equation
⇒ Wronskian constant

$$\dot{x}_2 x_1 - \dot{x}_1 x_2 = \text{const.}$$

$$x_2\left(\ddot{x}_1 + \omega^2(t)\,x_1\right) = 0 \qquad \Rightarrow \frac{d}{dt}\left(x_2\dot{x}_1 - \dot{x}_2 x_1\right) = 0$$
$$x_1\left(\ddot{x}_2 + \omega^2(t)\,x_2\right) = 0$$

<u>but</u>
$$W(x_1, x_2) = (\mu_1\mu_2)^{-1}\, W\left(x_1(t+T), x_2(t+T)\right)$$

<u>i.e.</u> consider time translation by T

$$W(x_1, x_2) = \left(\mu_2^{t/T}\,\Pi_2\right)\left(\mu_1^{t/T}\,\Pi_1\right) - \left(\mu_1^{t/T}\,\Pi_1\right)\left(\mu_2^{t/T}\,\Pi_2\right)$$

$$= \left(e^{(\ln\mu_2)t/T}\,\Pi_2\right) e^{(\ln\mu_1)t/T}\,\Pi_1 - \left(e^{(\ln\mu_1)t/T}\,\Pi_1\right) e^{\ln\mu_2\,t/T}\,\Pi_2$$

⇒ ∴ $\boxed{\mu_1\,\mu_2 = 1}$

– Can also observe:
1) coeffs in oscillator real, so
$x(t)$ an integral → x^* a solution
⇒
2) μ_1, μ_2 same as μ_1^*, μ_2^*
<u>i.e.</u>

$$\mu_1 = \mu_2^*, \quad \mu_2 = \mu_1^* \qquad \underline{\text{or}} \qquad \mu_1 = \mu_1^*, \quad \mu_2 = \mu_2^* \Big\rangle \text{ both real}$$

I   II

if I, $\mu_1\mu_2 = 1 \Rightarrow \mu_1 = 1/\mu_1^*$, $\mu_2 = 1/\mu_2^*$ $\Rightarrow \underline{|\mu_1|^2 = |\mu_2|^2 = 1}$

(trivial)

if (II) $\mu_1 \mu_2 = 1$; μ_1, μ_2 real $\Rightarrow$

$$\Rightarrow \quad x_1(t) = \mu^{t/T}\Pi_1(t) \; , \quad x_2(t) = \mu^{-t/T}\Pi_2(t)$$

i.e. 1 increasing, 1 decreasing } solution $\Rightarrow$ [parametric instability

[N.B. Exponential, not secular growth]!
$\Rightarrow$ "true" instability is possible
$\rightarrow$ Some Calculation (as basic structure of the solution established).

Consider Mathieu's eqn:

$$\ddot{x} + \omega_0^2\left[1 + h\cos[(2\omega_0 + \epsilon)t]\right]x = 0$$

bounds on ϵ for instability?

For solution, SHO $\Rightarrow$

$$x = a\cos(\omega_0 t) + b\sin(\omega_0 t)$$

so, in spirit of multiple-time-scale P.T. (i.e. $\omega^2(t)$ enters via $h \ll 1$ $\Rightarrow$ expect slow time scale variation of coefficients)

$$x = \underbrace{a(t)}\cos[(\omega_0 + \epsilon/2)t] + \underbrace{b(t)}\sin[(\omega_0 + \epsilon/2)t]$$

coeffs become slowly varying

Plugging it in:

$$\ddot{x} = \left(a(t)\cos[(\omega_0+\epsilon/2)t]\right)^{\cdot\cdot} + \text{O.T.}$$

$\hookrightarrow$ other term

$$= -(\omega_0+\epsilon/2)^2 a(t)\cos[\,] - 2(\omega_0+\epsilon/2)\dot{a}(t)\sin[\,] + \ddot{a}\cos[\,] + \text{O.T.}$$

neglect $\ddot{a}$, $\ddot{b}$ as h.o. in slowness (recall amplitude eqn. deriv.)

$\Rightarrow$ ω_0^2 term, only

$$-(\omega_0+\epsilon/2)^2 a(t)\cos[\,] - 2\dot{a}(t)(\omega+\epsilon/2)\sin[\,]$$
$$-(\omega_0+\epsilon/2)^2 b(t)\sin[\,] + 2\dot{b}(t)(\omega+\epsilon/2)\cos[\,]$$
$$+\omega_0^2\left[a(t)\cos[\,] + b(t)\sin[\,]\right]$$
$$+\omega_0^2 h\cos((2\omega_0+\epsilon)t)\left[a(t)\cos[\,] + b(t)\sin[\,]\right]$$
$$= 0$$

Now, - neglect $O(\epsilon^2)$ terms $\Rightarrow$ only $\omega_0\epsilon$ term survives.

- observe $\cos\left[(\omega_0+\frac{\epsilon}{2})t\right]\cos\left[(2\omega_0+\epsilon)t\right]$

$$= \frac{1}{2}\cos\left[3(\omega_0+\epsilon/2)t\right] + \frac{1}{2}\cos\left[(\omega_0+\epsilon/2)t\right]$$

{fast oscillation $\rightarrow$ non-resonant} (i.e. expect h.o. in h) — Resonant with fundamental

Resonant contribution is interesting one, here

$$\Rightarrow$$

$$-\omega_0\epsilon\left(a(t)\cos[\,] + b(t)\sin[\,]\right)$$
$$-2\dot{a}(\omega_0+\epsilon/2)\sin[\,] + 2\dot{b}(\omega_0+\epsilon/2)\cos[\,]$$
$$+\frac{\omega_0^2}{2}h\left[a(t)\cos[\,] - b(t)\sin[\,]\right]$$
$$=0$$

Regrouping coeffs. $\cos[\,]$, $\sin[\,]$;

$$\sin[\,]\left(-2\omega_0\dot{a} - b\omega_0\epsilon - \omega_0^2 h b/2\right)$$
$$+\cos[\,]\left(2\dot{b}\omega_0 - a\epsilon\omega_0 + \tfrac{1}{2}h\omega_0^2 a\right) = 0$$

$$\Rightarrow$$

$$(2\omega_0)\dot{a} + (\omega_0\epsilon)b + \left(\frac{\omega_0^2}{2}h\right)b = 0$$
$$(2\omega_0)\dot{b} - (\omega_0\epsilon)a + \left(\frac{\omega_0^2 h}{2}\right)a = 0$$

$$\Rightarrow \boxed{\begin{cases}\dot{a} + (\epsilon/2)b + (\omega_0 h/4)b = 0\\ \dot{b} - (\epsilon/2)a + (\omega_0 h/4)a = 0\end{cases}}$$

Basic system of Eqns for Amplitude Variation

$a(t) = a_0 e^{st}$
$b(t) = b_0 e^{st}$

exponentially growing/damping solution

$\Rightarrow$

$$s\, a_0 + (\epsilon/2 + \omega_0 h/4)\, b_0 = 0$$

$$\left(-\frac{\epsilon}{2} + \frac{\omega_0 h}{4}\right) a_0 + s\, b_0 = 0$$

$$\therefore \quad \boxed{s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4} = \frac{1}{4}\left(\frac{\omega_0^2 h^2}{4} - \epsilon^2\right)}$$

$\Rightarrow$ Parametric instability criterion.

Growth rate

Observe:

- instability for:

$$\epsilon^2 = (\gamma - 2\omega_0)^2 < \frac{\omega_0^2 h^2}{4}$$

$\omega_0 \to$ fundamental
$\gamma \to$ parametric variation freq.

amplitude of variation

$h^2 > 4(\gamma - \omega_0)^2/\omega_0^2$

i.e. sufficiently close to resonance $\Rightarrow$ growth.

- for $(\gamma - 2\omega_0)^2 > \omega_0^2 h^2/4 \to$ oscillation

- amplitude of $\omega_0^2(t)$ variation sets proximity threshold

more generally, can show when $\overset{\text{integer}}{n}\gamma = 2\omega_0$
$\Rightarrow$ parametric resonance. Of course, higher
$n \Rightarrow$ resonance region $\sim h^n$

- with friction, find threshold for instability:

i.e. $$(\gamma - 2\omega_0)^2 < \left[\left(\frac{1}{2}h\omega_0\right)^2 - 4\alpha^2\right]$$

friction coeff.

i.e. P.I. growth must beat damping!
Friction raises required h.

- Pumping on swing

$\rightarrow$ "pumping" $\rightarrow$ change of I

$$\ddot{\theta} + \frac{mg\ell\theta}{I(t)} = 0$$

$$I(t) = I_0 + \epsilon I_1(t)$$

$$\ddot{\theta} + \frac{g}{\ell}\theta + \frac{\epsilon g}{\ell}\frac{\Delta I(t)}{I}\theta = 0$$
$$+ \alpha\dot{\theta}$$

need pump twice per cycle

c) Nonlinear Oscillators – Conservative

→ Here, concerned with $\left\{\begin{array}{l}\text{nonlinear}\\\text{conservative}\end{array}\right\}$ oscillator systems, usually small perturbations about/away from the SHO

Prototype: Duffing's Equation

$$\underbrace{\ddot{x} + \omega_0^2 x}_{SHO,\ \omega_0} + \epsilon x^3 = 0$$

ϵx^3: NL, anharmonic term

Observe:

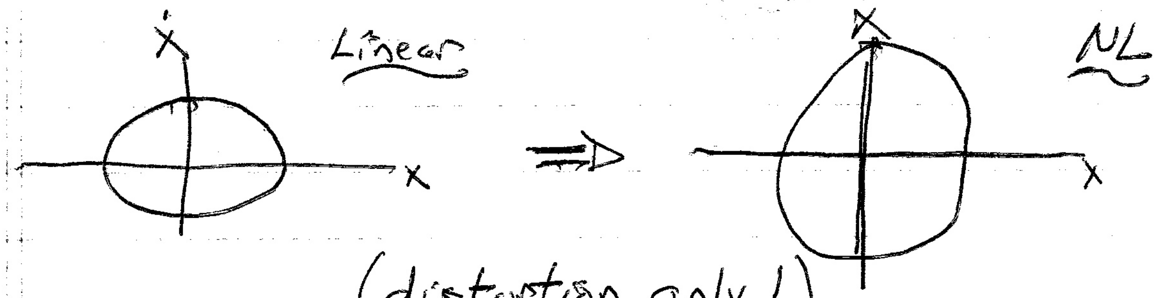
– $H = \frac{1}{2}\dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \epsilon \frac{x^4}{4}$

so $V(x) = \frac{1}{2}\omega_0^2 x^2 + \epsilon \frac{x^4}{4}$ $\quad (V(x)' > 0)$

– natural question to ask re: $\omega = \omega(\epsilon)$?
i.e. evolution of periodic/quasi-periodic orbits upon perturbation.

Now note: $V(x)$ bounded (from below) $\Rightarrow$ phase space / trajectory contours "contained"

i.e. ($\omega_0 = 1$)



(distortion only!)

so orbit must be bounded!

- A clue;

Observe: $\omega^2 = \frac{1}{2}\omega_0^2 \langle x^2 \rangle \Big/ \frac{\langle x^2 \rangle}{2}$

for SHO. $\left(\langle \, \rangle = \frac{1}{\tau}\int_0^{\tau} \right)$

$\Rightarrow$

$\therefore$ might expect, taking $\tau = 2\pi/\omega_0$

$$\omega^2 \cong \left(\frac{1}{2}\omega_0^2 \langle x^2 \rangle + \frac{\epsilon}{4}\langle x^4 \rangle \right) \Big/ \frac{\langle x^2 \rangle}{2}$$

(solve with SHO 'trial' fctn)

Now; $\langle x^4 \rangle = \langle a^4 (\cos \omega_0 t)^4 \rangle$

$$= a^4 \, (3/4)$$

$$\boxed{\omega^2 = \omega_0^2 + \frac{3}{4}\epsilon a^2}$$

$\rightarrow$ amplitude dependent frequency! (NL frequency shift)

$\leftrightarrow$ (nearly) correct result.

i.e. $\omega = \omega_0 \rightarrow$

$\omega = \omega(\omega_0, \epsilon, \underline{\underline{a}})$! $\rightarrow$ frequency becomes amplitude dependent.

- Systematics — Calculational Procedure

— expand in ϵ ¡ (surprise !?)

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0 \quad ; \quad x(0) = 0$$

$$x = x^{(0)} + \epsilon x^{(1)} + \dots$$

$\therefore$ $O(\epsilon^0)$: $\ddot{x}^{(0)} + \omega_0^2 x^{(0)} = 0$

$$x^{(0)} = a \sin(\omega_0 t)$$

$O(\epsilon^1)$: $\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\epsilon (x^{(0)})^3$

but $(x^{(0)})^3 = a^3 \sin^3 \omega_0 t$

$$= \frac{a^3}{4} (3 \sin \omega_0 t + \sin 3\omega_0 t)$$

$$\Rightarrow \quad \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\frac{\epsilon a^3}{4} (3 \sin \omega_0 t + \sin 3\omega_0 t)$$

$\left\{ \begin{array}{l} \text{resonant drive} \\ \Rightarrow \text{secularity !!} \end{array} \right.$

yields $x^{(1)} = \text{homog} + c\, t \cos \omega_0 t$

¿ is this physical ?

$$\Rightarrow -\omega_0^2 ct\cos\omega_0 t - 2c\omega_0\sin\omega_0 t$$

$$+\omega_0^2 ct\cos\omega_0 t = -\frac{\epsilon a^3}{4}(3\sin\omega_0 t)$$

$c = \dfrac{3\epsilon}{8\omega_0}$; $\epsilon \sim (\text{freq.})^2$, dimensionally

$$\therefore\ x = a\sin\omega_0 t + \frac{3}{8}\frac{\epsilon}{\omega_0}t\cos\omega_0 t + \ldots$$

secularity $\rightarrow$ $|x|$ diverges linearly in time

$\rightarrow$ Unphysical $\Rightarrow$ recall closed phase space trajectories!

What's going on?

Aside: A trivial example!?!!

$\ddot{x} + (1+\epsilon)^2 x = 0$; $x(0) = 0$, $\dot{x}(0) = 1$

if expand in ϵ;

$$\ddot{x} + x + 2\epsilon x + \epsilon^2 x = 0$$ (exactly solvable)

$\epsilon^{(0)}$; $\ddot{x}^{(0)} + x^{(0)} = 0$; $x^{(0)} = \text{sin}\,t$

$O(\epsilon)$; $\ddot{x}^{(1)} + x^{(1)} = -2\,x^{(0)}$

$\Rightarrow \; x^{(1)} = c\,t\left[\cos t\right] +$ homog:

$-2c\sin t - \cancel{ct\cos t} + \cancel{ct\cos t} = -2\sin t$

$c = 1$

- secularity !?!

This is clearly idiotic, since we all know

$x(t) = \sin\left[(1+\epsilon)t\right]$ trivially solves the problem!

Moral of this story:

② – to avoid secularity, must allow frequency shift; i.e. here $\omega = 1 \rightarrow \omega = 1+\epsilon$
gives all a warm, fuzzy .....

① – secularity results from breakdown of naive expansion in ϵ at long times, observe:

i.e. $x(t) = \sin\left[(1+\epsilon)t\right]$,

Taylor expansion in ϵ $\Rightarrow$ $\cong \sin t + \epsilon t \cos t$

↳ secularity → { artifact of expansion

→ The Fix:

- admit nonlinear frequency shift
  (i.e. method of Poincaré - Linstedt)
  Tip of Reductive P.T. ice-berg ....
- trick is to:

a) expand x, ω on equal footing

$$x = x^{(0)} + \epsilon x^{(1)} + \ldots$$
$$\omega = \omega^{(0)} + \epsilon \omega^{(1)} + \ldots$$

and use/choose $\omega^{(1)}$, etc. to cancel secularities

i.e. "solvability condition" ⟺ secularity removal.

i.e. $$\frac{d^2 x}{dt^2} + x + \epsilon x^3 = 0$$

usual long time behaviour ...

now! $$t = s\left(1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots\right)$$

time param.

and $$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

Improved Details

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0$$

now,

$$x = x^{(1)} + \underbrace{x^{(2)}}_{O(\epsilon)} + \underbrace{x^{(3)}}_{O(\epsilon^2)}$$

$$\omega = \omega_0 + \underbrace{\omega^{(1)}}_{O(\epsilon)\,(\text{shift})} + \omega^{(2)} + \ldots$$

and, $x_0 = a \cos \omega t$ ← note ω (not ω_0)

Now, re-write:

$$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\epsilon x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}$$

extra terms assure LHS $=0$, in lowest order.

$\Rightarrow$

$$\frac{\omega_0^2}{\omega^2}\left(-\omega^2 x^{(1)} + \ddot{x}_2\right) + \omega_0^2\left(x^{(1)} + x_2\right)$$

$$= -\epsilon\left(x^{(1)} + x^{(2)}\right)^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\left(-\omega^2 x^{(1)} + \ddot{x}_2\right)$$

$$\cancel{-\omega_0^2 x^{(1)}} + \ddot{X}_2 + \cancel{\omega_0^2 x^{(1)}} + \omega_0^2 X_2$$

$$= -\epsilon x^{(1)^3} - \left(-\omega^2 + \omega_0^2\right) x^{(1)}$$

$$- \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{X}_2$$

$\Rightarrow$

$$\ddot{X}_2 + \omega_0^2 \ddot{X}_2 = \underbrace{-\epsilon X^{(1)^3}}_{O(\epsilon)} + \underbrace{\left(\omega^2 - \omega_0^2\right)}_{O(\epsilon)} X^{(1)}$$

$$- \cancel{\left(1 - \frac{\omega_0^2}{\omega^2}\right)} \overset{\text{h.o.}}{\ddot{X}_2}$$

$$\omega^2 = \omega_0^2 + 2\omega_0\omega_1\epsilon \quad ; \quad \omega^2 - \omega_0^2 = 2\omega_0\omega_1\epsilon$$

$$x^{(1)^3} = a^3 \cos^3 \omega t = a^3 \left[\frac{1}{2} + \frac{\cos 2\omega t}{2}\right] \cos \omega t$$

$$= a^3 \left[\frac{\cos \omega t}{2} + \frac{1}{2}\left(\frac{1}{2}\right)\left(\cos 3\omega t + \cos \omega t\right)\right]$$

$$= a^3 \left[\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t\right]$$

$$\ddot{x}_2 + \omega_0^2 \dot{x}_2 = -\epsilon\left[a^3\left(\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right)\right] + 2\omega_0\omega_1\epsilon a\cos\omega t$$

Now to dodge the secularity, need cancel all resonant terms on RHS!

$\Rightarrow$

$$\ddot{x}_2 + \omega_0^2 \dot{x}_2 = -\epsilon\left[\cos\omega t\right]\left[\frac{3}{4}a^3 - 2\omega_0\omega_1 a\right] - \epsilon\frac{a^3}{4}\cos 3\omega t$$

$$\therefore \quad \boxed{\omega_1 = \frac{3}{8}a^2/\omega_0}$$

and

$$\boxed{\omega = \omega_0 + \epsilon\left(\tfrac{3}{8}a^2/\omega_0\right) + \cdots}$$

NL frequency shift!

and crank ⇒

$$x^{(2)} = \frac{-1}{2}\left(G\,a^3/16\omega_0^2\right)\cos 3\omega t.$$

Point:

- need get frequencies correct to avoid unphysical resonances, secularities...
- frequency correction ⇒ NL frequency shift

N.B.: compare:

- exact: $\omega_1 = 3/8\; a^2/\omega_0$
- rough: $\omega_1 = 3/4\; a^2/\omega_0$.

Moral: use frequency shift to eliminate secularity causing term on RHS.

so:

$$\frac{d^2}{ds^2}\left(X_0+\epsilon X_1+\epsilon^2 X_2+\cdots\right)+\left[\left(X_0+\epsilon X_1+\epsilon^2 X_2+\cdots\right)\right.$$

$$\left.+\epsilon\left(X_0+\epsilon X_1+\epsilon^2 X_2+\cdots\right)^3\right]\left(1+\epsilon\omega_1+\epsilon^2\omega_2+\cdots\right)^2=0$$

$O(\epsilon^0)$: $\quad \dfrac{d^2 X_0}{ds^2}+X_0=0$

$O(\epsilon^1)$: $\quad \dfrac{d^2 X_1}{ds^2}+X_1+X_0^3+2\omega_1 X_0=0$

$O(\epsilon^2)$: $\quad \dfrac{d^2 X_2}{ds^2}+X_2=-3X_0^2X_1-2\omega_1\left(X_1+X_0^3\right)-\left(\omega_1^2+2\omega_2\right)X_0$

etc.

Now,

$O(\epsilon^0)$: $\quad \dfrac{d^2 X_0}{ds^2}+X_0=0$

$$X_0=a\cos(s+\overset{\text{phase}}{\phi})$$

$O(\epsilon^1)$:

$$\frac{d^2x_1}{ds^2} + x_1 = -x_0^3 - 2\omega_1 x_0$$

$$= -a^3 \cos^3(s+\phi) - 2\omega_1 a \cos(s+\phi)$$

etc.

But

$$\cos^3(s+\phi) = \cos(s+\phi)\left[\frac{1}{2} + \frac{1}{2}\cos[2(s+\phi)]\right]$$

$$= \frac{1}{2}\cos(s+\phi) + \frac{1}{2}\cos(s+\phi)\cos[2(s+\phi)]$$

$$= \frac{\cos(s+\phi)}{2} + \frac{1}{4}\cos[3(s+\phi)] + \frac{1}{4}\cos[(s+\phi)]$$

$$= \frac{3}{4}\cos(s+\phi) + \frac{1}{4}\cos(3(s+\phi))$$

$\Rightarrow$

$$\frac{d^2x_1}{ds^2} + x_1 = -a^3\left(\underbrace{\frac{3}{4}\cos(s+\phi)}_{3}\overset{1}{} + \overset{2}{\frac{1}{4}\cos(3(s+\phi))}\right)$$

$$-2\omega_1 a \cos[(s+\phi)]$$

so

①, ③ $\sim \cos(s+\phi)$

resonates with RHS $\Leftrightarrow$
will drive secularity.

② $\sim \cos[3(s+\phi)]$

non-secular drive $\rightarrow$ harmless

$$\frac{d^2 x_1}{ds^2} + x_1 = \left[-\frac{3}{4}a^2 - 2\omega_1\right] a\cos(s+\phi)$$

$$+ \frac{1}{4}\cos(3(s+\phi))$$

so $\boxed{\omega_1 = -\frac{3}{8}a^2 \quad \text{removes secularity!!}}$

$t = s(1 - \frac{3}{8}\epsilon a^2 + \cdots)$

$\boxed{\omega = \omega_0\left[1 + \frac{3}{8}\epsilon a^2 + \cdots\right]}$

NL frequency shift!

i.e.

$s = t/1 - \frac{3}{8}\epsilon a^2$

$\cong t(1 + 3/8\,\epsilon a^2)$

and $x_1 = \frac{1}{32} a^3 \cos[3(s+\phi)]$

→ Frequency shift recovers guesstimate

→ example 2 – Nonlinear Klein Gordon Eqn.

KG eqn:

$$\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m'^2\phi = 0$$

(Scalar Field)

what physical system does this describe?

→ dispersion relation:

$$\omega^2 = c_0^2 k^2 + m'^2$$

(cold plasma wave)

$$\Rightarrow \mathcal{L} = \frac{\dot\phi^2}{2c_0^2} - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{m^2\phi^2}{2}$$

for nonlinearity:

$$U = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{m^2\phi^2}{2} + \frac{\alpha\phi^4}{4}$$

$$\Rightarrow \boxed{\frac{\partial^2\phi}{\partial t^2} - c_0^2\frac{\partial^2\phi}{\partial x^2} + m^2\phi = -\alpha\phi^3}$$

physics?

linear soln:

$$\omega^2 = c_0^2 k^2 + m^2$$

→ look for wave train solutions!

$$\phi = \phi(x - ct) = \phi(\theta)$$

sneaky → converts to ODE problem.

so, for NL problem

$$(c^2 - c_0^2)\,\phi'' + m^2\phi = -\alpha\phi^3$$

∴ expect nonlinearity will produce nonlinear phase velocity shift!

$$\Rightarrow \quad c = c^{(0)} + a^2 c_2 + \cdots$$

$$\phi = a\phi_1 + a^3\phi_3 + \cdots$$

} by correspondence with Duffing ($a \leftrightarrow \epsilon$)

n.b. here $\phi = \phi(\theta) \leftrightarrow$ wave train solution
∴ only parameter is $\underline{\underline{c}}$.

$$\Rightarrow \left((c^{(0)} + a^2 c_2)^2 - c_0^2\right)\left[a\phi_1'' + a^3\phi_3''\right] + m^2\left[a\phi_1 + a^3\phi_3\right] = -\alpha\left[(a\phi_1) + a^3\phi_3\right]^3 = 0$$

$$\left(c^{(0)2} - c_0^2\right)\phi_1'' + m^2\phi_1 = 0 \qquad O(a)$$

$$\left(c^{(0)2} - c_0^2\right)\phi_3'' + m^2\phi_3 = -2c^{(0)}c_2\phi_1'' - \alpha\phi_1^3 \qquad O(a^3)$$

now, $O(a)$:

$$\phi_1 = \cancel{\phi_0}^{1} \cos k\theta \qquad \text{(contained in } \theta\text{)}$$

$$c^{(0)2} = c_0^2 + \frac{m^2}{k^2}$$

$O(a^3)$:

$$\left((c^{(0)})^2 - c_0^2\right)\phi_3'' + m^2\phi_3$$

$$= -2c^{(0)}c_2 (\cos k\theta)'' - \alpha (\cos k\theta)^3$$

$$= 2c^{(0)}c_2k^2\cos k\theta - \alpha\cos k\theta\left[\frac{1}{2} + \frac{1}{2}\cos 2\theta k\right]$$

$$= 2c^{(0)}c_2k^2\cos k\theta - \frac{\alpha}{2}\cos k\theta + \frac{\alpha}{2}\frac{1}{2}\cos 3k\theta - \frac{\alpha}{2}\frac{1}{2}\cos k\theta$$

$$= 2c^{(0)}c_2k^2\cos k\theta - \frac{3}{4}\alpha\cos k\theta + \frac{\alpha}{4}\cos 3k\theta$$

so, to kill secularity:

$$2c^{(0)} c_2 k^2 = \frac{3}{4}\alpha$$

$$c_2 = \frac{3\alpha}{8c^{(0)}k^2}$$

$$\therefore \quad c = c^{(0)} + a^2 c_2$$

$$= \left(c_0^2 + \frac{m^2}{k^2}\right)^{1/2}\left[1 + \frac{3\alpha a^2}{8k^2 c^{(0)2}}\right]$$

$$\boxed{c = \left(c_0^2 + \frac{m^2}{k^2}\right)^{1/2}\left[1 + \frac{3a^2\alpha}{8(c_0^2k^2 + m^2)}\right]}$$

nonlinear speed change

shift

Key Point: ⇒ By expanding ω in ϵ, method of Poincaré and Lindstedt introduces additional degrees of freedom, so one can remove secularity order-by-order in P.T.

⇒ essence of NL oscillator is NL frequency shift, i.e.

{Pf. of asymptotics

$$\omega = \omega_0 + \frac{3\epsilon}{8\omega_0} a^2$$

) Forced Anharmonic Oscillator – Mode Jumping

– here, consider next step in development

⇒ forced Duffing's eqn.

$$\ddot{x} + \underbrace{2\lambda\dot{x}}_{\text{friction}} + \omega_0^2 x + \underbrace{\alpha x^2}_{\alpha=0,\text{ initially}} + \beta x^3 = \frac{f_{ext}}{m}$$

issue: combination $\begin{cases} \text{resonance} \\ \text{forcing} \\ \text{NL} \end{cases}$

recall, for linear forced SHO; (see 31)

$$a^2 = f^2/4m^2\omega_0^2\,(\epsilon^2+\lambda^2)$$ is $\left\{\begin{array}{l}\text{amplitude}\\\text{equation}\end{array}\right.$

$\epsilon = \omega - \omega_{res}$; $\omega_{res} = \omega_0$ (trivial)

Then for NL system: $\omega_{res} = \omega_0 + \kappa a^2$
(near primary linear resonance) $\kappa = \frac{3}{8}\frac{\epsilon\beta}{\omega_0}$ $\rightarrow$ NL shift

$\therefore$

$$a^2 = f^2/4m^2\omega_0^2\left((\omega-\omega_0-\kappa a^2)^2+\lambda^2\right)$$

$$\left\{\begin{array}{l} a^2\left[(\epsilon-\kappa a^2)^2+\lambda^2\right] = f^2/4m^2\omega_0^2 \\ a(\epsilon)\ \text{relation} \end{array}\right.$$

$\Rightarrow$ cubic equation for a^2!! (3 roots $\rightarrow$ which?)

Observe:

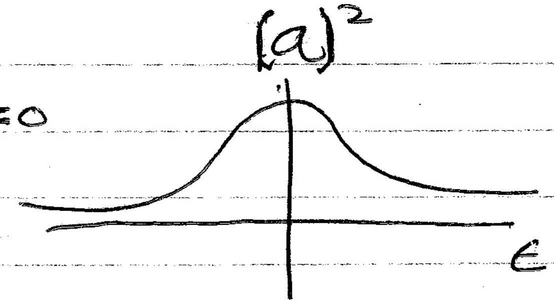
- addition of NL $\Rightarrow$ NL ω-shift $\Rightarrow$ non-trivial amplitude equation

- for $\omega = \omega_{res}$ $(\epsilon = \kappa a^2)$, $a^2 = f^2/4m^2\omega_0^2\lambda^2$
  $\therefore$ peak unchanged

but

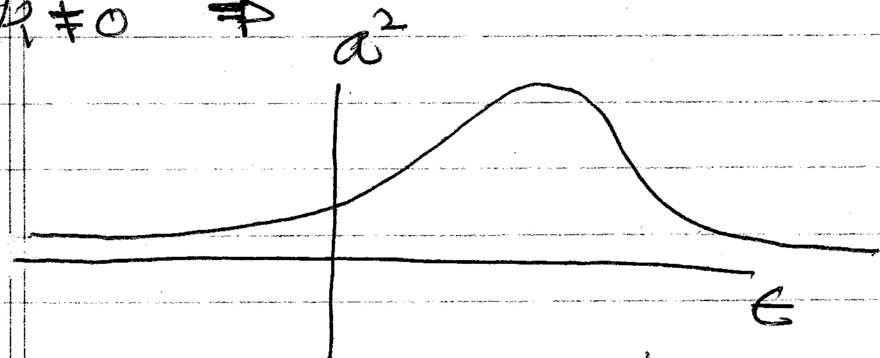
asymmetry induced in resonance curve!

i.e. $k=0$



i.e. Lorentzian (symmetric)

$k \neq 0 \Rightarrow$

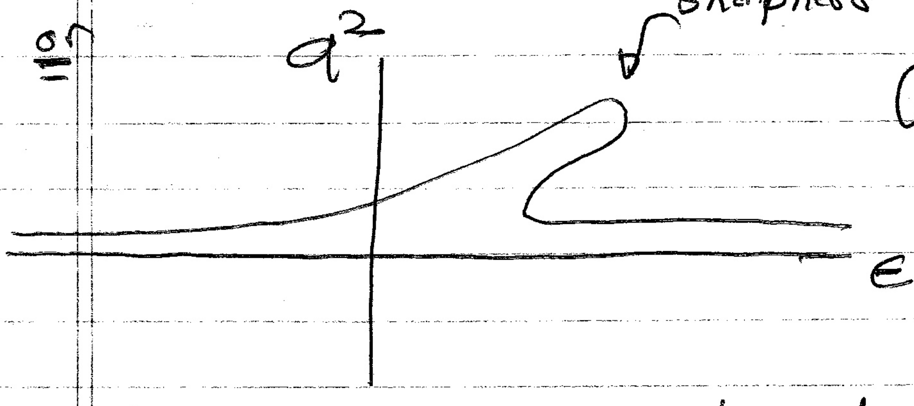


(asymmetric)

or

sharpness $\sim 1/\lambda \Rightarrow$ NL more impt.

(double valued!)



Can the resonance curve be double-valued?
— yes! → when $da^2/d\epsilon \to \infty$.

Now, amplitude equation: $a(\epsilon)$

$$a^2\left[(\epsilon - Ka^2)^2 + \lambda^2\right] = f^2/4m^2\omega_0^2$$

$$\Rightarrow a^2\left[\epsilon^2 - 2K\epsilon a^2 + (Ka^2)^2 + \lambda^2\right] = f^2/4m^2\omega_0^2$$

Can re-write amplitude equation as:

$$F(a^2, \epsilon) = a^2\left[\epsilon^2 - 2\epsilon(Ka^2) + (Ka^2)^2 + \lambda^2\right] - \frac{f^2}{4m^2\omega_0^2} = 0$$

so, for $da^2/d\epsilon$

on curve (defn.)

$$dF = 0 = (\partial F/\partial a^2)\, da^2 + (\partial F/\partial \epsilon)\, d\epsilon$$

$$\Rightarrow \quad \frac{da^2}{d\epsilon} = \frac{-(\partial F/\partial \epsilon)}{(\partial F/\partial a^2)}$$

$\partial F/\partial a^2 = 0$ for $\begin{cases} \text{double valuedness} \\ \text{2 roots} \end{cases}$ → though $\partial F/\partial a^2 = 0$ is coalescence pt. of two

$$F = K^2(a^2)^3 - 2\epsilon K(a^2)^2 + a^2(\epsilon^2 + \lambda^2) - \text{const}$$

$$\partial F/\partial a^2 = 3K^2(a^2)^2 - 4\epsilon K a^2 + (\epsilon^2 + \lambda^2) = 0$$

if $\quad x = Ka^2$

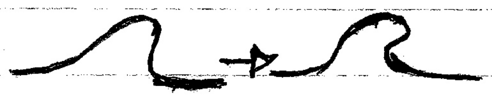
$$\partial F/\partial a^2 = 0 = 3x^2 - 4\epsilon x + (\epsilon^2 + \lambda^2)$$

$$\Rightarrow \quad x = \frac{4\epsilon}{6} \pm \frac{1}{6}\left(16\epsilon^2 - 12(\epsilon^2 + \lambda^2)\right)^{1/2}$$

$$= \frac{2}{3}\epsilon \pm \frac{1}{3}\left(\epsilon^2 - 3\lambda^2\right)^{1/2}$$

so $\kappa a^2 = \frac{2}{3}\epsilon \pm \frac{1}{3}(\epsilon^2 - 3\lambda^2)^{1/2}$

∴ → double valued-ness at $\begin{cases} \text{inflection pt.} \\ \text{2-root coalescence} \end{cases}$



i.e. when $\epsilon^2 = 3\lambda^2$, $\Rightarrow \kappa a^2 = 2\epsilon/3$

in terms external force magnitude ($a(\epsilon)$ relation)

$$f^2 = 4m^2\omega_0^2 a^2\left[\epsilon^2 - 2\epsilon\kappa a^2 + (\kappa a^2)^2 + \lambda^2\right]$$

$$\begin{cases} \kappa a^2 = \frac{2}{3}\epsilon = \frac{2\sqrt{3}}{3}\lambda \\ \epsilon = \sqrt{3}\lambda \end{cases}$$

and plugging into $f^2 \Rightarrow$

$$\boxed{f_{crit}^2 = 32m^2\omega_0^2\lambda^3/3\sqrt{3}\,|\kappa|}$$

$\begin{cases} \text{i.e. } f > f_{crit} \\ \text{required} \\ \text{for inflection} \end{cases}$

→ for peak value of amplitude

$$\partial F/\partial\epsilon = 0 \qquad (\Leftrightarrow da^2/d\epsilon = 0)$$

$\Rightarrow 2\epsilon - 2\kappa a^2 = 0 \Rightarrow \epsilon = \kappa a^2$

$\Rightarrow \omega - \omega_0 = \kappa a^2$, usual!

i.e. peak is at resonance (here $\omega = \omega_0 + Ka^2$), as usual.

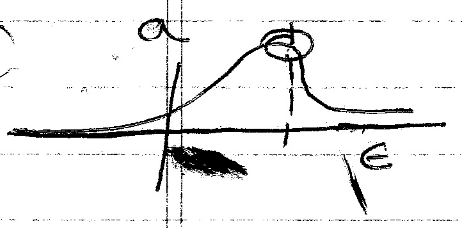
→ What's Going On?

- for $f^2 > f_{crit}^2 = 32 m^2 \omega_0^2 \lambda^3 / 3\sqrt{3}\,|K|$

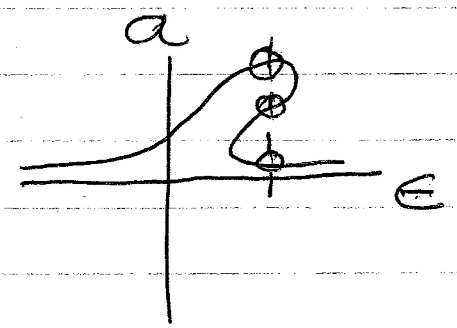
⇒ inflection occurs → bifurcation

↓

⇒ 1 root → 3 roots

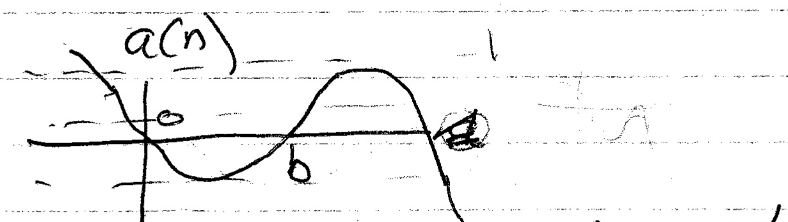


→



- of 3 roots ⇒ 2 stable
1 unstable

i.e. demonstration



$$\frac{\partial n}{\partial t} = f(n)$$

$$f(n) = \underbrace{-n(b-n)(n-1)}_{a(n)} + S$$

$$\frac{\partial n}{\partial t} = 0 \Rightarrow S = +n(b-n)(n-1)$$

↓

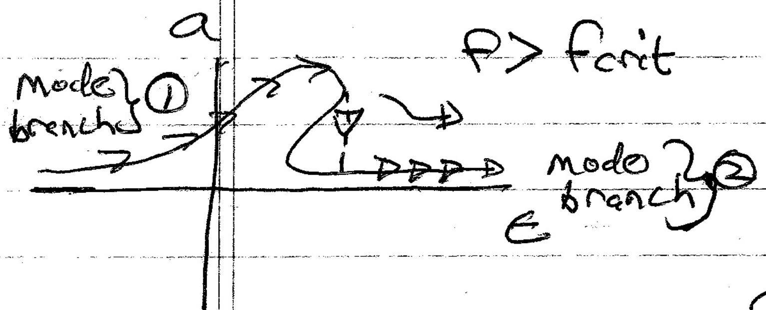
control ⇒ 3 or 1 roots, depending on S

but $n = n_{sol} + \delta n$

$$\frac{\partial \delta n}{\partial t} = \delta n \, f'(n_{sol}) \qquad f' \gtrless 0 \rightarrow \begin{cases} \text{instability} \\ \text{stability} \end{cases}$$

i.e. $f' > 0$ → unstable root (1)
$f' < 0$ → stable roots (2)

– 'bifurcation' occurs → jump between 2 stable branches (bifurcation ↔ inflection criterion)

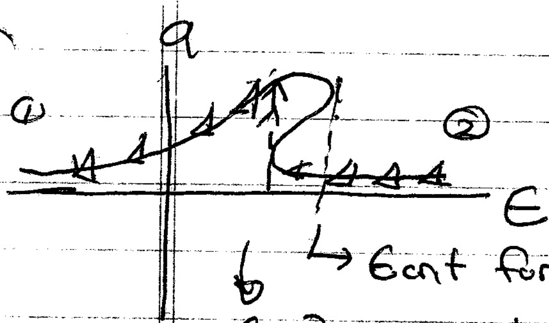


$\left\{ \begin{array}{l} a(f, \epsilon) \rightarrow \text{surface} \\ \text{catastrophe} \end{array} \right.$

$\epsilon > \epsilon_{crit}$
$f > f_{crit}$ ⇒ jump from $\left\{ \begin{array}{l} \text{branch} \\ \text{mode ①} \end{array} \right.$ to $\left\{ \begin{array}{l} \text{branch} \\ \text{mode ②} \end{array} \right.$

– system exhibits "hysteresis"

i.e. consider reversal of evolution from ② → ①, i.e.



$f > f_{cr}$

↳ ϵ_{crit} for forward transition

ϵ_{crit} for back transition

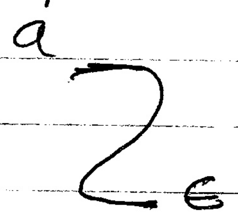
i.e.

$\epsilon_{fwd} > \epsilon_{back}$!

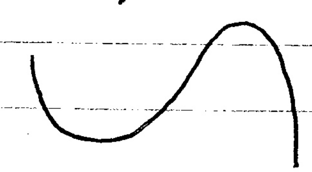
system tends to 'hang' in mode ②

→ driven Duffing oscillator is classic example of "S-curve" type bifurcation

i.e.



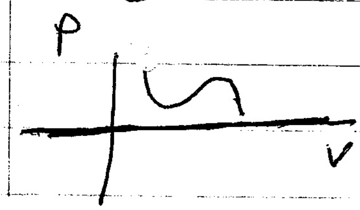
or



S-curve ⇒ bi-stable system with unstable root in between, yielding transitions or mode-jumping

⇒ akin phase transition
①, ② ⟷ 2 phase

S curve ⟷ p V curve



mode jumping ⟷ first order transition

→ Now,
- have examined impact of NL on resonance phenomena, at primary/linear resonance
- but, is this the whole story?

⇒ NL-induced resonance phenomena?! → {new resonance physics}

Now, re-insert αx^2 term:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x + \underset{\substack{\uparrow \\ \text{quadratic} \\ \text{NL}}}{\alpha x^2} + \beta x^3 = \frac{f_{ext}}{m}$$

here $\omega_{res} = \omega_0 + \kappa a^2$ — why $O(\alpha^2)$? → $x^{(2)}x^{(1)}$ beat

$$\kappa = \left(\frac{3\beta}{8\omega_0} - \underbrace{\frac{5\alpha^2}{12\omega_0^3}}_{\substack{\text{shift contribution,} \\ \text{due quadratic}}}\right)$$

→ derive

Why α^2?

- observe $-\alpha(x^{(0)})^2 \to \frac{1}{2}\cos(2\omega_0 t)$ > non-resonant drives to $O(a^2)$

⇒

- lowest secular contribution from αx^2 nonlinearity is in $x^{(3)}$ equation

here power a, not ϵ!