Special Topic: Higher Dimensions, Sympletic Structures and the Poincaré–Cartan Invariant

Observe:

→ Theory of canonical transformations, action-angle variables, adiabatic invariants rests on:

(1 deg freedom) invariance: $\oint_C p\,dq = \int_A dp\,dq$

i.e. phase area (phase volume) conservation.

→ what of higher dimensions?

Key: Hamiltonian EOM → symplectic

Punch line: ① $\oint \underline{p}\cdot d\underline{q} = \int_{A_r} \left(\sum dp \wedge d\underline{q}\right)$, conserved

sum: oriented areas

$\underline{p} \equiv$ gen. mom. vector arb. dimensions

$\underline{q} \equiv$ gen. coord vector arb dimensions

Poincaré relative invariant (abbreviated action)

② For Hamiltonian system:

$$\frac{d}{dt}\left(\oint_{A_i} d\underline{p} \wedge d\underline{q}\right) = 0$$

akin Liouville Thm. but not identical

(i.e. sum of oriented areas is const.).

Need develop: — wedge product
— higher-d Stokes thm

(n.b.: will <u>not</u> prove everything)

→ Physical Idea:

$$\boxed{\oint \underline{p} \cdot d\underline{q} \equiv \text{conserved } \underline{\text{phase space circulation}} \rightarrow \text{property of phase space flow}}$$

∴

What is a circulation invariant and what does it mean?

Consider a fluid:

NSE

$$\boxed{\rho\Big(\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v}\cdot\underline{\nabla}\,\underline{v}}_{\text{advection}}\Big) - \underbrace{\eta\nabla^2\underline{V}}_{\text{viscous transport of momentum}} = \underbrace{-\underline{\nabla} P}_{\text{pressure gradient}}}$$

<u>i.e.</u>

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} - \underbrace{\nu}_{\text{kinematic viscosity}}\nabla^2\underline{V} = -\frac{\underline{\nabla} P}{\rho}$$

<u>now</u>

$$\underline{v}\cdot\underline{\nabla}\,\underline{v} = +\underline{\nabla}\left(\frac{v^2}{2}\right) + \underline{\omega}\times\underline{v}$$

$$\underbrace{\underline{\omega}}_{\text{vorticity}} = \underline{\nabla}\times\underline{v}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\underline{V} = -\frac{\underline{\nabla}p}{\rho} - \underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{V}\times\underline{\omega}$$

for isentropic flow:

$$\partial s/\partial t + \underline{V}\cdot\underline{\nabla}S = 0 \qquad S \equiv \text{entropy}$$

so

$$dE = dQ - p\,dV$$
$$= T\,dS - p\,dV$$

$$\underset{\text{enthalpy}}{dH} = T\,dS - p\,dV + d(pV) \qquad \left(\text{Legendre transform}\right)$$

$$= T\,dS + V\,dp$$

so for const. entropy

$$dH = T\cancel{dS} + \frac{dp}{\rho}$$

$$\Rightarrow \quad \nabla H = \frac{1}{\rho}\underline{\nabla}p$$

$$\frac{\partial \underline{V}}{\partial t} = -\underline{\nabla}H - \underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{V}\times\underline{\omega} + \nu\nabla^2\underline{V}$$

$$\boxed{\begin{aligned}\frac{\partial \underline{\omega}}{\partial t} &= \underline{\nabla}\times(\underline{V}\times\underline{\omega}) + \underline{\nabla}\times\nu\nabla^2\underline{\underline{\omega}}\\ &= \underline{\nabla}\times\underline{V}\times\underline{\omega} + \nu\nabla^2\underline{\omega}\end{aligned}}$$

define a circulation

$$\boxed{I_c = \oint_C \underline{v} \cdot d\underline{l} = \int_A d\underline{a} \cdot \underline{\omega}}$$

$\oint \underline{v} \cdot d\underline{l}$

$\int d\underline{a} \cdot \underline{\omega}$

Important to think of circulation as both
→ line integral velocity
→ area integral of vorticity projection

$$\frac{d}{dt} I_c \underset{\nu \to 0}{=} 0$$

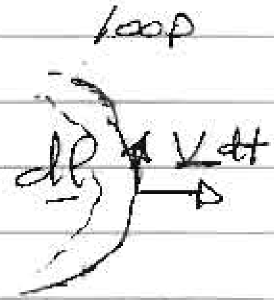
i.e. absent viscosity, circulation conserved

Key point: Invariance of circulation linked to area integration, changes in vorticity & motion of loop

Proof

$$\frac{d}{dt} \int_A d\underline{a} \cdot \underline{\omega} = \int_A d\underline{a} \cdot \frac{\partial \underline{\omega}}{\partial t} + \int_A \underline{\omega} \cdot \frac{d\underline{a}}{dt}$$

$$= \int_A d\underline{a} \cdot (\underline{\nabla} \times \underline{v} \times \underline{\omega}) + \int_A \underline{\omega} \cdot \frac{d\underline{a}}{dt}$$



$$= \oint_C d\underline{l} \cdot \underline{v} \times \underline{\omega} + \oint_C \underline{\omega} \cdot \frac{\underline{v} \times d\underline{l}\, dt}{dt}$$

defines $d\underline{a}$

i.e. $d\underline{a} =$ [redacted] $\underline{v}\, dt \times d\underline{l}$

$$\frac{dI_c}{dt} = \oint_c d\underline{\ell} \cdot \underline{v} \times \underline{\omega} - \oint_c d\underline{\ell} \cdot \underline{v} \times \underline{\omega}$$

$$= 0 \checkmark$$

⇒ Kelvin's Circulation Theorem!

$$\frac{d}{dt} I_c = 0 \quad , \quad \underline{I_c} = \oint \underline{v} \cdot d\ell = \int d\underline{a} \cdot \underline{\omega}$$

for $\nu = 0$

So, – have shown the 'Mother of All Circulation Theorems"

– need prove the counter-part for Poincaré's Invariant

i.e. $\frac{d}{dt}\left(\oint p \cdot d\underline{q} \right) = 0$ ¿

Note:

i.) Another approach (less edifying):

for Euler eqn:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = \frac{d\underline{v}}{dt} = -\underline{\nabla}H$$

Now

$$I_c = \oint_c \underline{v}\cdot d\underline{\ell}$$

so

$$\frac{dI_c}{dt} = \oint_c \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint_c \underline{v}\cdot\frac{d\underline{\ell}}{dt}$$

$$= -\oint_c \underline{\nabla}H\cdot d\underline{\ell} + \oint_c \underline{v}\cdot d\underline{v}$$

$$= \cancel{-\oint d H} + \cancel{\oint_c d(v^2/2)}$$

$$= 0$$

circulation conserved.

•.) An Example

→ Vortex tube stretching.

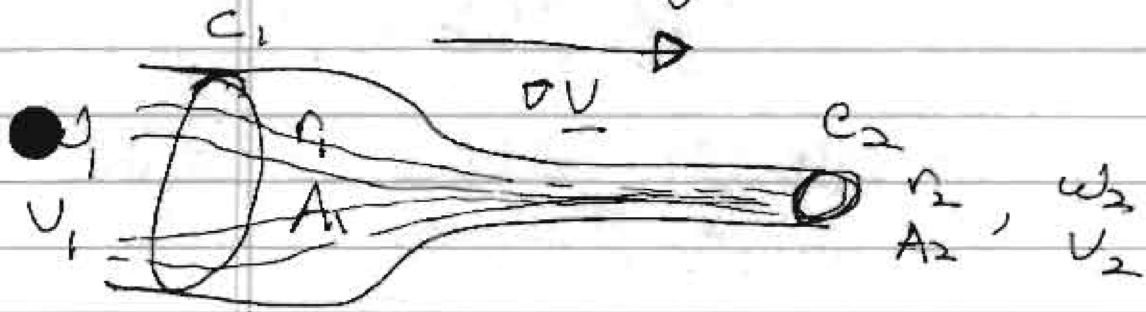
Consider incompressible flow:

$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{\omega})$$

$$= -\underline{v} \cdot \underline{\nabla} \underline{\omega} + \underline{\omega} \cdot \underline{\nabla} \underline{v}$$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \underline{\nabla} \underline{v}$$

Consider a straining flow:



incompressible
⇒ stretching
conserves
volume.

Now K-Thm ⇒

$$\omega_1 A_1 = 2\pi r_1 v_1 = \omega_2 A_2 = 2\pi r_2 v_2$$

so $\frac{v_2}{v_1} = \frac{r_1}{r_2}$ ⇒ velocity on small scale increased by stretching event.

$\frac{\omega_2}{\omega_1} = r_1^2/r_2^2$ ⇒ argument for enstrophy production in stretching
⇒ forward cascade in 3D turbulence.

→ Eine kleine Geometry

(a) Contravariant and Covariant Vectors

Consider, for arbitrary dimensions:

$\underline{a} = (a_1, \ldots a_n)$

where each $a_i = a_i(x_1, \ldots x_n)$
so $a_i = a_i(\underline{x})$

Now, consider transformation $x \to y$
such that $y = y(x)$

→ $\underline{a}$ contravariant if transforms like:

$$\bar{a} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} a_j$$

transform of $\underline{a}$ (new variables)

example: $d\underline{x} = (dx_1, \ldots dx_n) \equiv d\underline{x}_n$

infinitesimal distance

$$d\underline{y} = (dy_1(\underline{x}) \dots dy_n(\underline{x}))$$

$$dy_i = \sum_j \frac{\partial y_i}{\partial x_j} dx_j \quad \rightarrow \text{contravariant}$$

$\rightarrow \underline{b}$ covariant if transforms according to:

$$\bar{b}_i = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} b_j$$

example: gradients of scalar field are covariant

if $b_i = \partial S / \partial x_i$ $\quad \underline{b} = \underline{\nabla} S$

$\underline{b}(\underline{x})$!

$$\bar{b}_i = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial S}{\partial x_j}$$

$$= \frac{\partial S}{\partial y_i}$$

$$= \sum_{j=1}^{n} \frac{\partial S}{\partial x_j} \frac{\partial x_j}{\partial y_i} \quad \checkmark \quad = \sum_{j=1}^{n} b_j \frac{\partial x_j}{\partial y_i}$$

obviously, since $\underline{p} = \frac{\partial S}{\partial \underline{q}} = \nabla_{\underline{q}} S$

$\Rightarrow$ generalized momenta vectors are covariant

(b) Differential Forms

→ Differential 1-form ω^1 defined as:

$$\omega^1 = b_1 dx_1 + b_2 dx_2 + \dots b_n dx_n$$

where $\underline{b}_i = \underline{b}_i(\underline{X})$ is covariant

$$\omega^1 = \sum_{i=1}^{n} b_i dx_i = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i \frac{\partial x_i}{\partial y_j} dy_j$$

in terms of y

$$= \sum_{j=1}^{n} \bar{b}_j dy_j$$

$$\bar{b}_j = \sum_i b_i \frac{\partial x_i}{\partial y_j}$$

also

$$\omega^1 = \sum_{i=1}^{n} b_i dx_i = \sum_{j=1}^{n} \bar{b}_j dy_j$$

$\therefore$ $\rightarrow$ ω' invariant under transformation of variables

(c)

Now, Poincare invariant

$$I_p = \oint \underline{p} \cdot d\underline{q}$$

is related to 1-form:

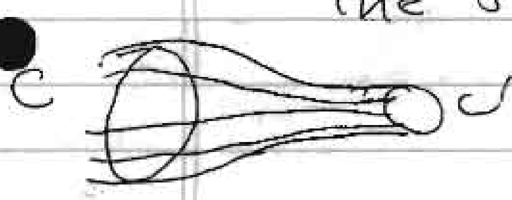
$$\omega' = \underline{p} \cdot d\underline{q} = \sum_{i=1}^{n} p_i dq_i$$ (arbitrary dimensions)

N.B.: $\omega^{(1)}$ as 1-form is consequence of covariance of $\underline{p}$, which is in turn a consequence of relation between momenta and action, i.e. $\underline{p} = \partial S / \partial \underline{q}$.

(d) More on 1-forms:

1-forms obey a "Kelvin's Circulation Theorem"

i.e. consider a bundle or tube of trajectories $\Rightarrow$ circulation is the same for all closed curves which encircle the same set of trajectories!



i.e.

$$\boxed{\oint_{c} \omega^{1} = \oint_{c'} \omega^{1}}$$

invariance of circulation (not proof here.)

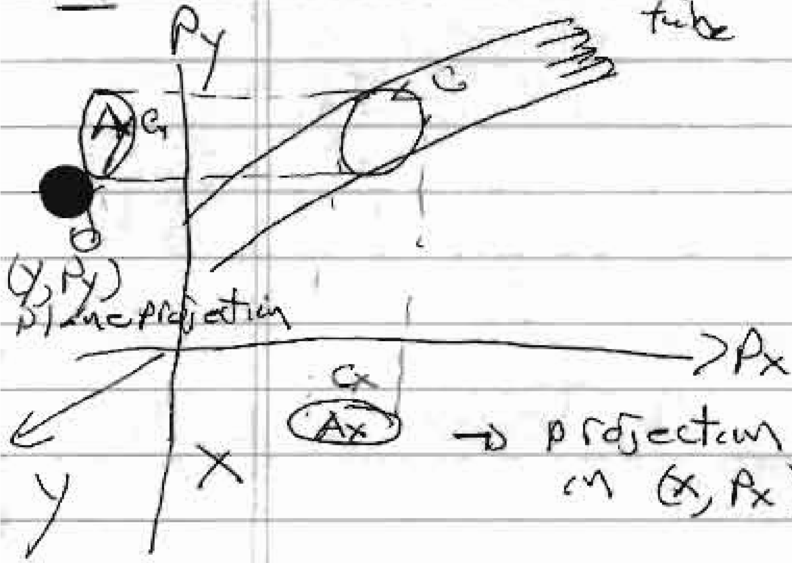
where:

$$\oint_{c} \omega^{1} = \oint_{c} \sum_{i=1}^{n} p_i \, dq_i = \sum_{i=1}^{n} \oint_{c_i} p_i \, dq_i$$

integral

sum of projected line integrations → i plane

i.e.



→ projection in (x, p_x) plane

(can skip → later)

of course, it is evident that:

$$\oint \omega^{1} = \sum_{i=1}^{n} \oint_{c_i} p_i \, dq_i = \sum_{i=1}^{n} \int_{A_i} dp_i \, dq_i$$

[skip above bundle]

i.e.

$$\boxed{\frac{d}{dt} \oint \omega_{\text{Poin.}} = \frac{d}{dt} \oint_{c} p \, dq = 0}$$

prove later, but evident from above.

(D) 2-forms:

Can construct differential 2-forms from products — "wedge products" — of one forms.

For $\begin{pmatrix}\text{exterior} \\ \text{wedge}\end{pmatrix}$ products:

$$d\underline{x}_i \wedge d\underline{x}_j = -d\underline{x}_j \wedge d\underline{x}_i$$ (anti-symmetry) (like cross product)

$$d\underline{x}_i \wedge d\underline{x}_i = 0$$

and

$b \rightarrow$ scalar

$$b_i \wedge d\underline{x}_j = dx_j \wedge b_i = b_i d\underline{x}_j$$

$$d\underline{x}_i \times (b_i d\underline{x}_j) = b_i d\underline{x}_i \wedge d\underline{x}_j$$

n.b. → can build from basis vectors and rules for basis wedge products

$$\underline{e}_1 \wedge \underline{e}_2 = \underline{e}_3 \ , \ \underline{e}_1 \wedge \underline{e}_3 = -\underline{e}_2 \ \ldots$$

→ obvious similarity to cross products...

Now consider 1 forms:

$$a_1 = b_1 dx_1 + b_2 dx_2$$
$$a_2 = c_1 dx_1 + c_2 dx_2$$

$$\theta_1 \wedge \theta_2 = (b_1\, dx_1 + b_2\, dx_2) \wedge (c_1\, dx_1 + c_2\, dx_2)$$

$$= (b_1 c_2 - b_2 c_1)\, dx_1 \wedge dx_2$$

$$= -\theta_2 \wedge \theta_1$$

$\theta_1 \wedge \theta_2$ is $\rightarrow$ a 2-form
$\rightarrow$ 2-form denoted by $\omega^{(2)}$.

Why are 2 forms interesting? $\rightarrow$ Phase area (volume) conservation!

i.e. Consider 2D, and change of variables:

$$x_1 = x_1(y_1, y_2) \qquad \underline{x} \rightarrow \underline{y}$$
$$x_2 = x_2(y_1, y_2)$$

$$d\underline{x}_1 = \frac{\partial x_1}{\partial y_1} dy_1 + \frac{\partial x_1}{\partial y_2} dy_2$$

$$dx_2 = \frac{\partial x_2}{\partial y_1} dy_1 + \frac{\partial x_2}{\partial y_2} dy_2$$

so, for transformation of 2-form:

$$dx_1 \wedge dx_2 = \left(\frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} - \frac{\partial x_2}{\partial y_1} \frac{\partial x_1}{\partial y_2} \right) dy_1 \wedge dy_2$$

$$= J(x_1, x_2; y_1, y_2)\, dy_1 \wedge dy_2$$

$$= \frac{\partial (x_1, x_2)}{\partial (y_1, y_2)}\, dy_1 \wedge dy_2$$

thus, if $x_1 \to p$ $\quad$ $y_1 \to P$

$x_2 \to q$ $\quad$ $y_2 \to Q$

and variables (properly) canonical

$$\Rightarrow \quad dp \wedge dq = dP \wedge dQ$$

and, in general:

$$\sum_{i=1}^{n} dp_i \wedge dq_i = \sum_{i=1}^{n} dP_i \wedge dQ_i$$

and can generalize to

$$\omega^{2n} = \prod_{i=1}^{n} dp_i \wedge dq_i \equiv \text{phase volume}$$

Yet more:

→ can obtain 2-forms from 1-forms by exterior differentiation

i.e. $\underset{\text{1 form}}{\omega^1} = \sum_{i=1}^{n} b_i \, dx_i$

⇒

$$d\omega^1 = d\left(\sum_{i=1}^{n} b_i \, dx_i\right) = \underbrace{\sum_{i=1}^{n} db_i \wedge dx_i}_{\text{2 form}}.$$

example:

$$\omega^1 = b_1(x_1, x_2)\, dx_1 + b_2(x_1, x_2)\, dx_2$$

then

$$d\omega^1 = db_1 \wedge dx_1 + db_2 \wedge dx_2$$

$$= \left(\frac{\partial b_1}{\partial x_1} dx_1 + \frac{\partial b_1}{\partial x_2} dx_2\right) \wedge dx_1 + \left(\frac{\partial b_2}{\partial x_1} dx_1 + \frac{\partial b_2}{\partial x_2} dx_2\right) \wedge dx_2$$

$$= \left(\frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

then this implies:

$$\omega^1 = f\,dx + g\,dy$$

$$\omega^2 = d\omega^1 = df \wedge dx + dg \wedge dy$$

$\Rightarrow$

$$\oint_C \omega^1 = \iint_A \omega^2$$

$\Rightarrow$ Stokes Thm.

Now, $\omega^1 = p\,dq$

then $\displaystyle \oint p\,dq = \iint_A dp \wedge dq$

and can immediately generalize to

$$d\left(\sum_{i=1}^{n} p_i\,dq_i\right) = \sum_{i=1}^{n} dp_i \wedge dq_i$$

and

$$\oint_C \sum_{i=1}^{n} p_i\,dq_i = \sum_{i=1}^{n} \iint_{A_p} dp_i \wedge dq_i$$

projected area.

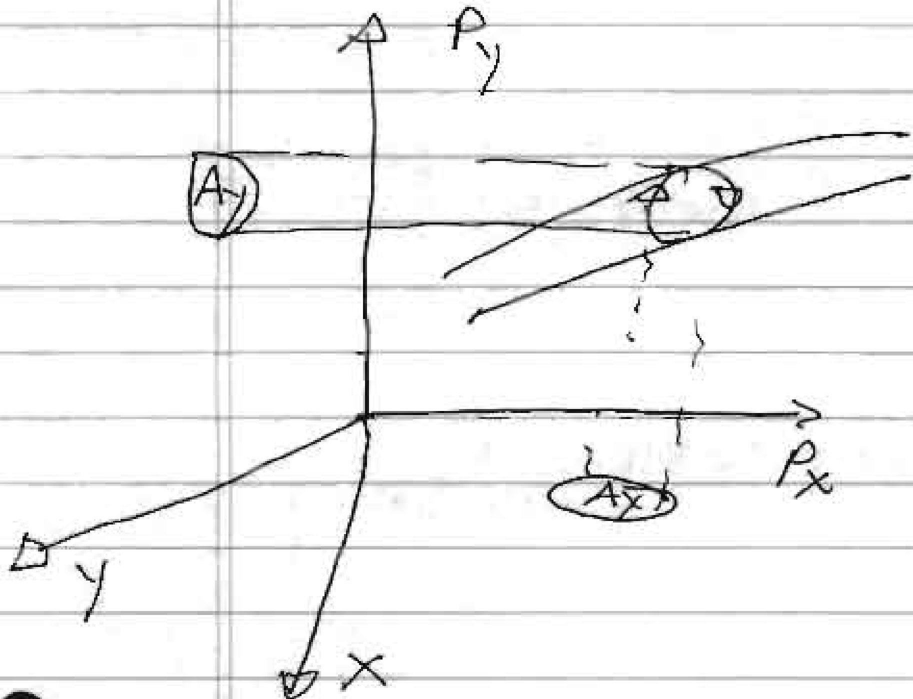
i.e. have a key result:

$$\oint \sum_{i=1}^{n} p_i \, dq_i = \oint \underline{p} \cdot d\underline{q}$$

Poincare invariant

$$= \sum_{i=1}^{n} \iint_{A_{proj.}} dp_i \wedge dq_i$$

↳ area from projecting C onto (p_i, q_i) planes.



P Punchline

Now: Poincare's Theorem $\rightarrow$ show $d\underline{q} \wedge d\underline{p}$ conserved, directly

Define: $\omega = d\underline{q} \wedge d\underline{p}$

$\downarrow$
oriented
phase area
$\Rightarrow$ 2-form

and, of course:

$$(\dot{\underline{q}}, \dot{\underline{p}}) = (\partial H/\partial \underline{p}, -\partial H/\partial \underline{q})$$

Then:

$$\frac{d\omega}{dt} = \frac{d}{dt}(d\underline{q} \wedge d\underline{p})$$

$$= (d\dot{\underline{q}} \wedge d\underline{p}) + (d\underline{q} \wedge d\dot{\underline{p}})$$

$$= (d(\partial H/\partial \underline{p}) \wedge d\underline{p}) + (d\underline{q} \wedge d(-\partial H/\partial \underline{q}))$$

$$= \left(\left(\frac{\partial^2 H}{\partial \underline{q} \partial \underline{p}}\right) d\underline{q} + \left(\frac{\partial^2 H}{\partial \underline{p}^2}\right) d\underline{p}\right) \wedge d\underline{p}$$

$$+ d\underline{q} \wedge \left(-\left(\frac{\partial^2 H}{\partial \underline{q}^2}\right) d\underline{q} - \frac{\partial^2 H}{\partial \underline{p} \partial \underline{q}} d\underline{p}\right)$$

but $d\underline{q} \wedge d\underline{q} = 0$
$d\underline{p} \wedge d\underline{p} = 0$

so

$$\frac{d\omega}{dt} = \left(\frac{\partial^2 H}{\partial \underline{q}\partial \underline{p}}\right) d\underline{q} \wedge d\underline{p}$$

$$- \left(\frac{\partial^2 H}{\partial \underline{q}\partial \underline{p}}\right) d\underline{q} \wedge d\underline{p}$$

$$= 0 \checkmark$$

Thus:

→ Hamiltonian flow conserves oriented phase area ("phase volume")

→ Hamiltonian flows are symplectic

→ $\oint \underline{p} \cdot d\underline{q}$ circulation is conserved.

i.e have

$$\boxed{\oint \underline{p} \cdot d\underline{q} = \oint \sum_{i=1}^{n} p_i dq_i = \sum_{i=1}^{n} \iint_{A_p} dp_i \wedge dq_i}$$

and all conserved

N.B.: Liouville's Thm appears as conservation of 2n-form

$$\omega^{2n} = \prod_{i=1}^{n} dp_i \wedge dq_i$$

exercise → show this!

∞

→ can construct C.T.'s from invariance $\oint \underline{p} \cdot d\underline{q}$

→ can extend to:

$p d\underline{q} - H dt$ for time dependent case

∴ all C-T theory, adiabatic invariant theory survive in higher dimensions.

→ for Action-Angle variables:

- consider system with n IOMs

∴

- $I_i = \frac{1}{2\pi} \oint_{\gamma_i} p \cdot dq$ ≡ Action variables!
  $i = 1, n$

  γ_i ↑ i-cycle, which conserves I_i → n indep. cycles - 1 per IOM.

- angles → phase on cycle

i.e. $\underline{\dot{\theta}} = \dfrac{\partial H}{\partial \underline{I}}$

References for this:

V.I. Arnold, "Mathematical Methods of Classical Mechanics"
- a classic, <u>not</u> easy

D.D. Holm, "Geometrical Mechanics" Vol. I
- good, easier than Arnold, but still very formal

M. Tabor, "Chaos and Integrability in Nonlinear Dynamics"
- easier but not as deep

Kelvin's Thm in fluids;

L. Landau, E.M. Lifshitz, "Fluid Mechanics"