Introduction to Synchronization

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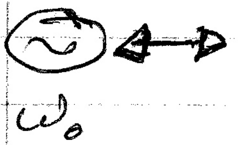
I.) Synchronization – Frequency Locking

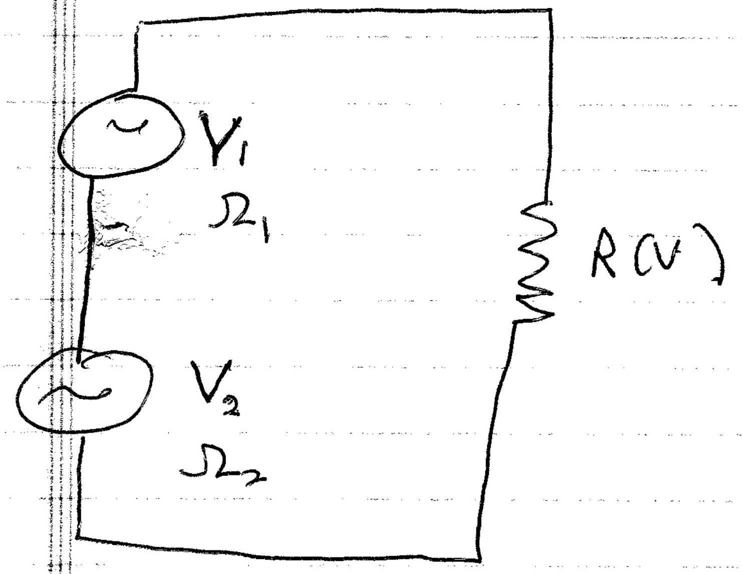
Loosely put, synchronization is concerned with understanding how, when, why:

- an external force can 'entrain' a nonlinear oscillator, i.e.



$F_{ext}(\omega)$ response at ω, ω_0 ?

- one oscillator 'entrains' another, i.e.



i.e. of:

$V_{\frac{1}{2}} = V_{\frac{1}{2}}(I)$,

possibility of synchronization or chaos exists ?

- network of coupled oscillators synchronize, de-synchronize, break into chaos or 'turbulence' etc. ?

- how network of coupled oscillators responds to noisy excitation?

(*) Central focus of study of synchronization:

→ phase dynamics, via progressive examples.

* — basic example

i.) { Mode Locking
Frequency Locking
Entrainment } of Single Nonlinear

Oscillator with External Periodic Force.

a.) weak forcing ↔ perturbative approaches

i.e. weak < phase equation ↔ method of averaging (Landau-Stuart, Complex Ginzburg-Landau Equations, etc.)

b.) Strong Forcing ↔ semi-quantitative insights from dynamical systems theory.

iii)

→ Simplest Example of Non-Trivial Phase Dynamics – the Limit Cycle

- Consider M-dimensional autonomous system:

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}) \qquad \begin{cases} \underline{x} = 1, \ldots M \\ f(\underline{x}) \text{ independent time} \end{cases}$$

"Limit Cycle" or "Self-Sustained Oscillation" ~ is stable periodic solution s/t

$$x(t+T) = x(t)$$

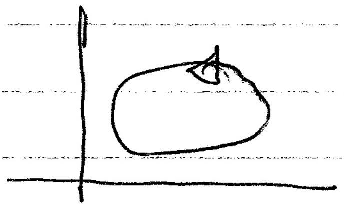
$\Leftrightarrow$

- generally useful to distinguish between:

i.) "weak" or non-attracting limit cycle, which is not an attractor in phase space

i.e.

$$dq/dt = \alpha p$$
$$dp/dt = -\alpha q$$



closed orbit in phase space of Hamiltonian system.

$\Downarrow$

ii.) "strong" or attracting limit cycle, which is a phase space attractor classic example – in addition to V-d-P

i.e. CGL or Landau – Stuart Equation

$$\frac{dA}{dt} = (1 + i\eta)A - (1 + i\alpha)|A|^2 A$$

(amplitude equation)

linear growth

linear frequency

non-linear saturation

nonlinear frequency shift

then $A = Re^{i\theta}$ → phase

amplitude

[n.b.: amplitude and phase necessary due comlex equation]

$$dR/dt = R(1-R^2)$$

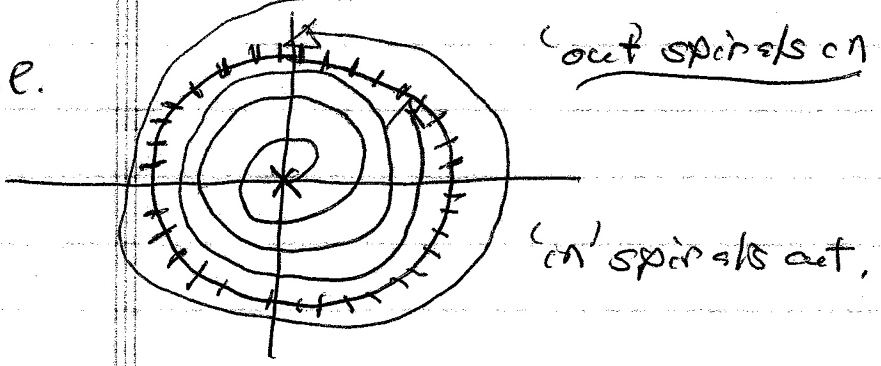
$$d\theta/dt = \eta - \alpha R^2$$

Fixed 'points':
$R = 0, 1$

$R=0$ → center (unstable)

$R=1$ → limit cycle → stable attractor

i.e.



N.B. {Limit cycle necessarily must have unstable fixed point at its center.

skp

Another classic example: Van-der-Pol's Equation, c.c.

{nonlinear triode
simple heart model

$$\ddot{x} - 2\mu\dot{x}(1-\beta x^2) + \omega_0^2 x = 0$$

$$\begin{tikzcd} & & \text{nonlinear} \\ 2\mu\dot{x}(1-\beta x^2) \arrow[d] & & \\ \text{dissipation} \arrow[d] \arrow[uurr, dash] \arrow[drr, dash] & & \\ \text{cycle solutions} & & \text{negative } (x < \beta^{-1/2}) \arrow[r] \arrow[d] & {} \\ & & \text{instability} & \end{tikzcd}$$

in phase plane:

$$\dot{x} = y \equiv F(x,y)$$

$$\dot{y} = 2\mu y(1-\beta x^2) - \omega_0^2 x \equiv G(x,y)$$

convenient to change variables:

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

so

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$= \cos F(r\cos\theta, r\sin\theta) + \sin\theta\, G(r\cos\theta, r\sin\theta)$$

and

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} \qquad (\text{"angular momentum"})$$

$$\dot{\theta} = \frac{\cos\theta}{r} G(r\cos\theta, r\sin\theta) - \frac{\sin\theta}{r} F(r\cos\theta, r\sin\theta)$$

$\beta \equiv 1$

so $\dot{r} = -\mu(r^2\cos^2\theta - 1)\, r\sin^2\theta$

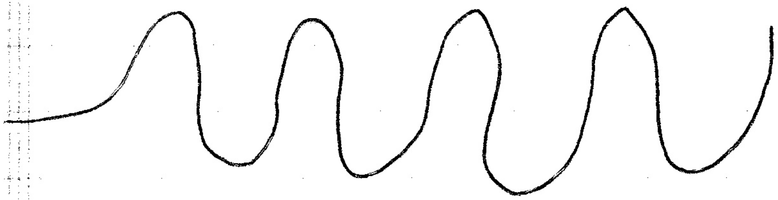
$\dot{\theta} = -1 - \mu(r^2\cos^2\theta - 1)\cos\theta\sin\theta$

can treat perturbatively, via method of averaging
- see Drazin "Nonlinear Systems" for reference, discussion
- also treated in notes
- essentially angle average (→ fast time scale)

Physics: H.O. $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$

VdP $\lambda \to -\mu(1-x^2)$ < nonlinear, negative

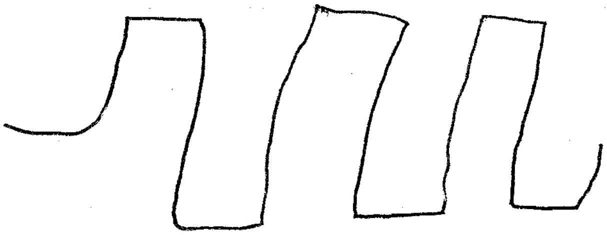
$\Rightarrow \mu \ll 1 \to$ NL oscillation



'rise' → $x < 1$

'fall' → $x > 1$

$\mu \gg 1 \to$ sawtooth cycle



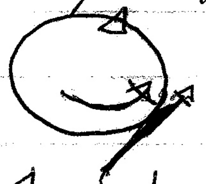
"sequence of 'rise, fall' transitions"

⊛

In general, limit cycle has property that:

- $\frac{d\phi}{dt} = \underbrace{\omega_0}_{\text{natural frequency of ``self-sustained oscillation''}}$

- (strong) limit cycle is attractor, but phase neutrally stable



∴ 1 Lyapunov exponent $h = 0$ (corresponds to motion along attractor)

$\Rightarrow$

$\rightarrow$ phase stable, but <u>not</u> asymptotically stable

- so if consider small perturbations on oscillator:

<u>i.e.</u> $\frac{d\underline{x}}{dt} = \underbrace{\underline{f}(\underline{x})}_{\omega_0} + \epsilon \underbrace{\underline{p}(\underline{x}, t)}_{\omega \neq \omega_0}$

as cycle is attractor:

- excursion (induced by perturbation) $\perp$ to limit cycle <u>necessarily</u> <u>small</u>

but - excursion along on on cycle can be large

$\Rightarrow$ consistent with phenomenology of phase jumps, etc.

B.) Basics of Phase Dynamics

Note: - r, Θ description $\Rightarrow$ r_c, Θ describe dynamics on cycle

(*) - seek extend description, i.e.

$\Rightarrow$ dynamics of phase off, but near cycle. $\rightarrow$ $\boxed{\text{phase field}}$ $\Rightarrow$ crucial

$\downarrow$ extended

$\Rightarrow$ Isochrones

- math speak:

consider mapping $\underline{\Phi}(\underline{x}): x(t) \rightarrow x(t+T_0)$

if $\underline{x}_*$ on cycle, the set of points attracted to $\underline{x}_*$ defines $M-1$ dimensional hypersurface, called isochrone hypersurface.

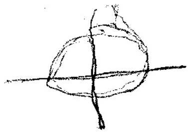
- in physical terms; on isochrones:

a.) flow takes 1 period → next
b.) flow along isochrones rotates attractor at same rate as cycle ω_0

i.e. recall CGL: (Complex Ginzburg-Landau)

$$dR/dt = R(1-R^2)$$
$$d\theta/dt = 1 - \alpha R^2$$

{isochrones are curves



then, for initial condition R_0, θ_0:

$$R(t) = \left[1 + \left(\frac{1-R_0^2}{R_0^2}\right) e^{-2t}\right]^{-1/2}$$

$$\theta(t) = \theta_0 + (1-\alpha)t - \frac{\alpha}{2} \ln\left(R_0^2 + (1-R_0^2)e^{-2t}\right)$$

$t \to \infty$

$$R(t) \to 1$$
$$\theta(t) = \theta_0 + (1-\alpha)t - \alpha \ln R_0$$

This suggests:

$$\underbrace{\phi}_{\text{(extended) phase}}(R,\theta) = \underbrace{\theta - \alpha \ln R}_{\text{generalized phase definition to whole plane}}$$

i.e.
$$\frac{d\phi}{dt} = \frac{d\theta}{dt} - \alpha \frac{\dot{R}}{R}$$

$$= 1 - \alpha R^2 - \frac{\alpha}{R}\left(R - R^3\right)$$

$$= 1 - \alpha \; !$$

$$d\phi/dt = 1 - \alpha \quad \Rightarrow$$

for $\phi = \theta - \alpha \ln R$, phase ϕ rotates uniformly. !

as on attractor

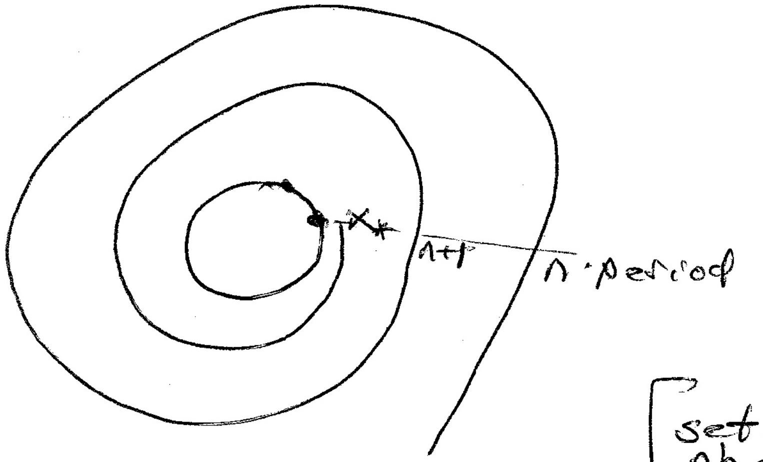
so, - isochrones are curves on which phase rotates uniformly

- observe isochrones here are spiral curves:

$$\theta - \alpha \ln R = \theta_0$$ (const phase)

$$R = \exp\left[(\theta - \theta_0)/\alpha\right]$$

i.e.



isochrones are spirals attracted to limit cycle

[set of isochrones → phase field]

essence of study of synchronization is study of phase dynamics.

⇒ Basic Theory of Phase Dynamics for Small Perturbations

Let $\phi(\underline{x}) \equiv$ phase in some neighborhood of an attracting limit cycle

$$\frac{d\phi(\underline{x})}{dt} = \omega_0$$

$$= \sum_k \frac{\partial \phi}{\partial x_k} \frac{dx_k}{dt}$$

$$= \sum_k \frac{\partial \phi}{\partial x_k} f_k(\underline{x}) \equiv \omega_0$$

Now, consider perturbations,

i.e. $$\frac{d\underline{x}}{dt} = f(\underline{x}) + \epsilon p(\underline{x},t)$$

$$= \sum_{\underline{n}} \frac{\partial\phi}{\partial x_n}\left(f_n(\underline{x}) + \epsilon p_n(\underline{x},t)\right)$$

Now, in perturbation theory,

l.o. $$\frac{d\underline{x}}{dt} = \sum_{\underline{n}} \frac{\partial\phi}{\partial x_n} f_n(x)$$

$\lim_{t\to\infty} \underline{x}(t) = \underline{x}_0$ → 'the' Limit cycle

Limit cycle ⇒ rotation frequency ω_0

$$\begin{cases} \underline{x}_0 = \underline{x}_0(\phi) \\ \phi = \phi_0 + \omega_0 t \end{cases}$$

1st order: $$\frac{d\phi}{dt} = \omega_0 + \epsilon Q(\phi,t)$$

ϕ → "extended" phase

ϕ rotates at ω_0 along isochrones

so $$\boxed{\frac{d\phi}{dt} = \omega_0 + \epsilon Q(\phi_0,t)}$$ phase dynamics equation

see next

where: $$Q(\phi, t) = \sum_k \frac{\partial \phi}{\partial x_k}(\underline{x}_0(\phi))\, p_k(\underline{x}_0(\phi), t)$$

$\downarrow$ [evaluate forcing along unperturbed orbits, i.e. $\phi = \phi_0 + \omega_0 t$

so to 1st order:

$$\rightarrow \boxed{\frac{d\phi}{dt} = \omega_0 + Q(\phi_0 + \omega_0 t, t)}$$

Shir

Eg: CGL, Again .....

in Cartesian coordinates:

$$\frac{dx}{dt} = x - \eta y - (x^2+y^2)(x - \alpha y) + \epsilon \cos \omega t$$

$$\frac{dy}{dt} = y + \eta x - (x^2+y^2)(y + \alpha x)$$

then, if: $$\phi = \tan^{-1}(y/x) - \frac{\alpha}{2} \ln(x^2+y^2)$$

$$\frac{d\phi}{dt} = \omega_0 + \epsilon \frac{\partial\phi}{\partial x}\cos\omega t$$

$$= \eta - \alpha - \epsilon\,(\alpha\cos\phi + \sin\phi)\cos\omega t$$

if $\tan\phi_0 = 1/\alpha$

$$\frac{d\phi}{dt} = \eta - \alpha - \epsilon(1+\alpha^3)^{1/2}\cos(\phi-\phi_0)\cos\omega t$$

- phase equation for ϕ in presence of small, periodic external force
- Can be treated in p.t. and more generally.

→ resume

Now, $Q(\phi,t) \equiv$ external force with 'own frequency' ω. Recall aim is to determine when external force 'entrains' oscillator.

→ periodic in ϕ, t

∴ Can write:

$$Q(\phi,t) = \sum_{l,k} q_{l,k}\, e^{ik\phi}\, \underset{\substack{\uparrow \\ \text{ext force}}}{e^{il\omega t}}$$

$$\frac{d\phi}{dt} = \omega_0 + Q(\underset{\uparrow}{\phi},t)$$

$$\phi = \phi_0 + \omega_0 t \quad \Rightarrow$$

$$Q(\phi, t) = \sum_{l,k} q_{l,k} \, e^{ik\phi_0} \, e^{i(k\omega_0 + l\omega)t}$$

so most important contribution to $Q(\phi, t)$ are those terms:

- yielding ~ d.c./steady Q, which induce phase secularities
- $\boxed{\text{resonances: } k\omega_0 + l\omega \cong 0}$

N.B.: Akin resonant, rational surfaces in torus $q = m/n \equiv q(r)$

Simple case: of $\omega \sim \omega_0 \Rightarrow k = -l$
(no loss generality) is dominant, so:

$$\frac{d\phi}{dt} = \omega_0 + \epsilon Q(\phi - \omega t)$$

$$Q = \sum_l q_l \, e^{il[\phi - \omega t]}$$
$$\equiv q(\phi - \omega t)$$

so natural to define: $\boxed{\psi = \phi - \omega t}$ phase variable

deviation from rotation
with forcing

$$\frac{d\psi}{dt} = \frac{d\phi}{dt} - \omega = \omega_0 + \epsilon g(\psi) - \omega$$

so

$$\boxed{\frac{d\psi}{dt} = -\nu + \epsilon g(\psi)}$$

Simple Phase Dynamics Equation

$\nu \equiv \omega - \omega_0 \equiv$ Frequency mismatch

$\epsilon g(\psi) \equiv$ forcing

(*)

N.B. - Competition between frequency mismatch of oscillator with entraining forcing (more generally: entrainee vs. entrainer) is essence of synchronization problem.

- more generally, mis-match vs. interaction strength generic to any nonlinear mode coupling problem:] *

i.e. – parametric oscillator instability ⟺

$$\gamma^2 = \frac{1}{4}\Big[\big(\underbrace{\tfrac{1}{2}\Delta\omega_0}_{\text{osc amplitude}}\big)^2 - \underbrace{\epsilon^2}_{\hookrightarrow\ \omega-\omega_0}\Big]$$

– forced Duffing bifurcation

$$b^2 = \frac{f^2/4m^2\omega_0^2}{\big[(\underbrace{\epsilon}_{\text{mis-match}} - \underbrace{kb^2}_{\to\ \text{NL shift}})^2 + \lambda^2\big]}$$

– 3 wave coupling in turbulence

$$\Theta_{\underline{k},\underline{p},\underline{q}} = i \Big/ \Big[\overbrace{(\omega_{\underline{p}} + \omega_{\underline{q}} - \omega_{\underline{k}})}^{\to\ \text{triad mis-match}} + i\underbrace{(\mu_{\underline{k}} + \mu_{\underline{p}} + \mu_{\underline{q}})}_{\hookrightarrow\ \text{NL broadening}}\Big]$$

triad resonance

$$\underbrace{\mu_k}_{\text{self-damping}} = \sum_{\underline{k}'} |(CC)_{\underline{k}'}|^2 \, \overbrace{|\tilde{V}_{\underline{k}'}|^2}^{\text{amplitude}} \, \underbrace{\Theta_{\underline{k}+\underline{k}'}}_{\text{triad coherence time.} \iff \text{mis-match}}$$

$$\boxed{\frac{d\psi}{dt} = -\nu + \epsilon g(\psi)}$$

1D dynamical system ⇒
parameters $\left\{ \begin{array}{l} \epsilon \\ \nu \end{array} \right.$

So, for synchronization / phase locking ⇒ seek:

- stable fixed points ✓

i.e. $\frac{d\psi}{dt} = 0 \Rightarrow \nu = \epsilon g(\psi_s)$

↳ ψ_{synch}.

and $\psi = \psi_s + \delta\psi$

$$\frac{d}{dt}\delta\psi = \epsilon g'(\psi_s)\,\delta\psi$$

⇒ $g'(\psi_s) < 0$ → stable fixed point

$g'(\psi_s) > 0$ → unstable fixed point

Obviously, $g'(\psi_s) < 0$ ⇒ stable fixed points
⇒ synchronized states

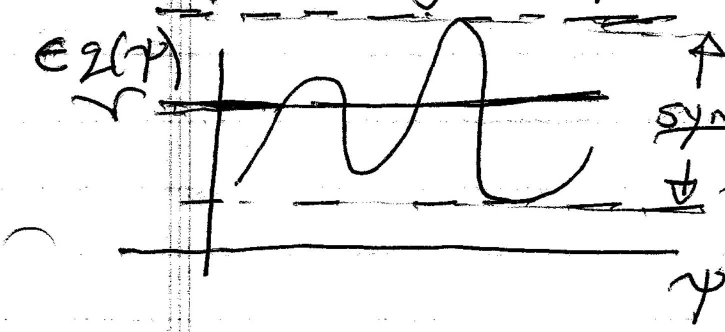
Note: - at $\psi = \psi_s$

$$\phi - \omega t = \psi_s$$

$$\phi = \psi_s + \omega t$$

↳ oscillator phase "synchs" to external force.

- in general



synchronization region $(\epsilon q)_{min} < r < (\epsilon q)_{max}$

- synchronization for $\epsilon q_{min} < r < \epsilon q(\psi)_{max}$.

- fixed points come in stable-unstable pairs (curve crossings)

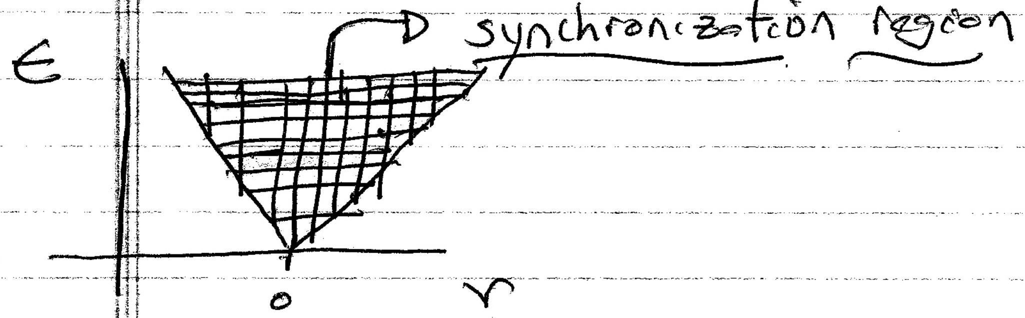
- except when pair disappears

i.e. synchronization transition (a bifurcation!), where stable and unstable points collide

∴ onset of synchronization ⇒ bifurcation

Synchronization Region

$\epsilon \xi_{min} < r < \epsilon \xi_{max}$ ⇒ boundaries are straight lines



- several pairs of fixed points can exist in synchronization region

⇒ several synchronized states possible, given mis-match, etc.

Now, if ν outside synchronization range:

$$\frac{d\psi}{dt} = -\nu + g(\psi) \qquad t = \int^{\psi} \frac{d\theta}{[\epsilon g(\psi) - \nu]}$$

⇒ gives $\psi(t)$, so $\phi = \omega t + \psi(t)$

→ - case of quasi-periodic motion, with two incommensurate periods.

- periods / frequencies

⇒ driver: $2\pi/\omega$, ω

⇒ "beat frequency" ≡ difference between observed oscillator frequency and external force frequency.

↓

effective frequency of $\psi = \phi - \omega t$

$$T_\psi = \left| \int_0^{2\pi} d\psi \Big/ \left(\epsilon \zeta(\psi) - \nu\right) \right| \quad \rightarrow \text{beat period}$$

$$\Omega_\psi = 2\pi / T_\psi \quad \rightarrow \text{beat frequency}$$

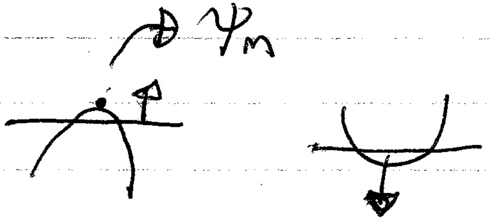
$$\text{so } \langle \dot\phi \rangle = \Omega = \omega + \Omega_\psi$$

time avg, on ω^{-1}.

actual observed frequency

Now, how does T_ψ, Ω_ψ etc. behave near synch transition point?

- near synchronization



fixed points collide....

- $\nu_{Max} = \epsilon \zeta_{max}$, then expanding:

$$\epsilon\zeta(\psi) - \nu = \epsilon\zeta(\psi) - \nu_{max} - (\nu - \nu_{max})$$

$$= \cancel{\epsilon\zeta(\psi_m)} + \cancel{\epsilon\zeta'(\psi_m)(\psi-\psi_m)} + \frac{1}{2}\epsilon\zeta''(\psi_m)(\psi-\psi_m)^2 - \cancel{\nu_{max}} - (\nu - \nu_{max})$$

so $$T_\psi \cong \int_0^{2\pi} d\psi \Big/ \left[\tfrac{1}{2}\epsilon\xi''(\psi_m)(\psi-\psi_m)^2 - (r-r_M)\right]$$

obviously, T_ψ dominated by contribution where $|\epsilon\xi(\psi) - r| \to 0$, i.e. ψ near bifurcation point!!

$$\Rightarrow T_\psi \simeq \int_{-\infty}^{+\infty} \frac{d\psi}{(r-r_M)} \left(1 \Big/ \left[\frac{1}{2}\frac{\epsilon\xi''(\psi_m)(\psi-\psi_m)^2}{(r-r_M)} - 1\right]\right)$$

$$\cong \left[\epsilon\,\xi''(\psi_{max})(r-r_M)\right]^{-1/2} \quad \#$$

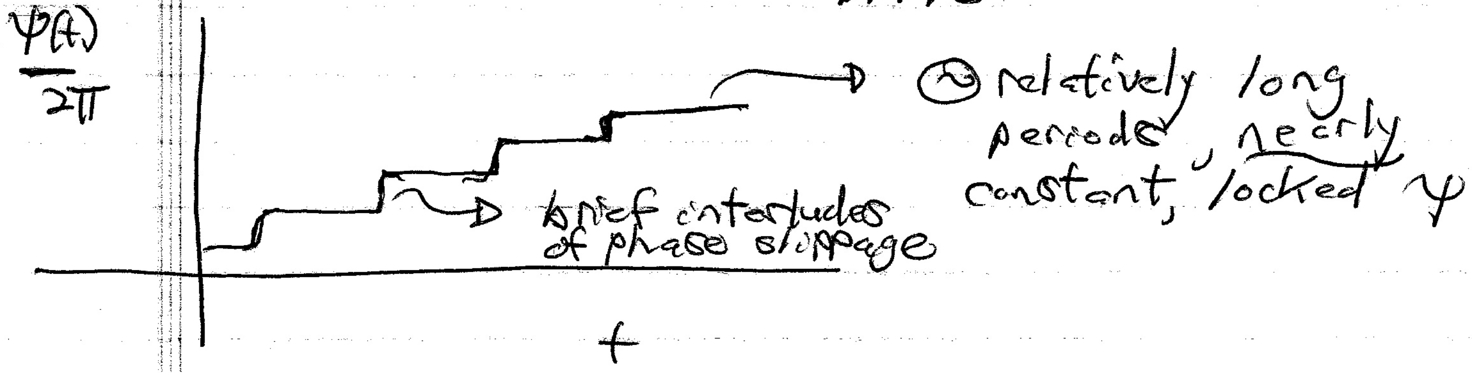
$$\Rightarrow \Omega_\psi \sim \left[\epsilon\,\xi''(\psi_{max})(r-r_M)\right]^{1/2} \rightsquigarrow$$ frequency of phase jumps (not evenly distr.)

$$\sim \sqrt{\epsilon}\,(r-r_M)^{1/2}$$

beat frequency, as $r \to r_M$ (from outside synch. region)

i.e. not surprisingly, beat frequency slows near bifurcation point ⇒ system spends long time near ψ_{max}

so ⇒ bottom line: $\psi = \phi - \omega t$ looks like:



i.e. trajectory:

- long periods of near synchronization, where phase nearly locked

interspersed between

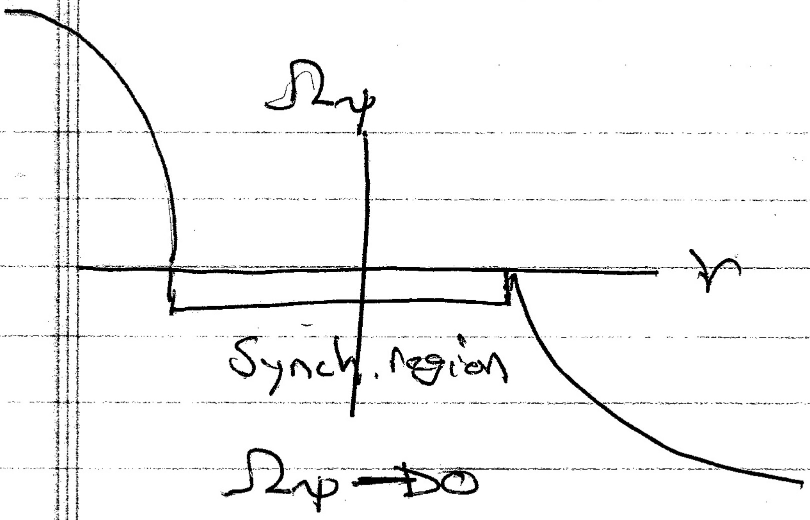
- short periods of rapid phase variation, or slips. Phase rotates by 2π

during slip ⟺ "phase slip"

- slip is much longer in duration than ω^{-1}

- transition to synchronization ⇒ time interval between slips increases!

i.e.



$|\Omega_\psi| \sim \sqrt{r - r_M}$ away from synch region.

→ time interval between slips increases approaching bifurcation point (unless time intervals of slips diverge there).

II.) Technical Aside: IMPORTANT

- Where does CGL come from?
- Why is it so *!ξ√X ubiquitous?

Consider nonlinear oscillator:

$$\underbrace{\ddot{x}+\omega_0^2 x}_{\text{oscillator}} = \underbrace{f(x,\dot{x})}_{\text{nonlinearity}} + \epsilon\, \underbrace{p(t)}_{\text{forcing} \to \text{frequency } \omega}$$

seek: $$x(t) = \tfrac{1}{2}\Big(\underbrace{A(t)}_{\text{amplitude}}\, e^{i\underbrace{\omega}_{\text{"entraining" frequency}} t} + c.c.\Big)$$

(not necessarily slow → i.e. phase jumps)

Now, then convenient to re-write:

$$\ddot{x} + \omega^2 x = (\omega^2 - \omega_0^2)x + f(x,\dot{x}) + \epsilon p(t)$$

<u>or</u> $\dot{x} = y$

$$\dot{y} = -\omega^2 x + (\omega^2 - \omega_0^2)x + f(x,y) + \epsilon p(t)$$

so <u>if</u> $y = \frac{1}{2}\left(i\omega A(t)\, e^{i\omega t} + c.c.\right)$

$(y = \dot{x})$

then: Amplitude Eqn ⟷ Complex A

$$* \quad \boxed{\underbrace{\dot{A}(t)}_{\text{Amplitude}} = \frac{e^{-i\omega t}}{i\omega}\left[\underbrace{(\omega^2-\omega_0^2)}_{\text{mis-match}}x + \underbrace{f(x,y)}_{\text{nonlinearity}} + \underbrace{\epsilon p(t)}_{\text{forcing}}\right]}$$

Now, as usual:

- interested in <u>slowest</u>, largest variations on RHS

  ⇒ isolate slowest terms

∴

- eliminate fast oscillations via averaging (akin method of averaging)

To Average:

- substitute x, y in terms $A(t)$ on RHS of $*$
- neglect oscillating terms (on ω scale)

$$\dot{A} = \frac{e^{-i\omega t}}{i\omega}\left[\overset{\textcircled{1}}{(\omega^2-\omega_0^2)x} + \overset{\textcircled{2}}{f(x,y)} + \overset{\textcircled{3}}{\epsilon p(t)}\right]$$

①:

$$\dot{A}_{\textcircled{1}}(t) = \int_0^{2\pi/\omega} d\tau \, \frac{e^{-i\omega\tau}}{i\omega}\left[(\omega^2-\omega_0^2)\left(\frac{1}{2}A(t)e^{i\omega\tau} + c.c.\right)\right]$$

$$= \frac{(\omega^2-\omega_0^2)}{2i\omega} A(t)$$

②:

$$A_{\textcircled{2}}(t) = \int_0^{2\pi/\omega} d\tau \, \frac{e^{-i\omega\tau}}{i\omega} f(x,y)$$

Now, $f(x,y)$ is nonlinear, in general,

so

$$f(x,y) = \sum_{m,n} C_{m,n} \left(A e^{i\omega t}\right)^{n} \left(A^{*} e^{-i\omega t}\right)^{m}$$

$\Rightarrow$

$$A_{\textcircled{2}} = \int_{0}^{2\pi} d\tau \, \frac{e^{-i\omega\tau}}{i\omega} \sum_{m,n} C_{m,n} \left(A e^{i\omega t}\right)^{n} \left(A^{*} e^{-i\omega t}\right)^{m}$$

only $n - m - 1 = 0$ don't vanish

$m = n - 1$

(extracts phase coherent piece of nonlinearity coherent with $e^{i\omega t}$)

Thus $A_{\textcircled{2}}$ must have form:

$$A_{\textcircled{2}} = \underbrace{g}_{\text{arbitrary (set by problem)}}(|A|^2) A$$

(i.e. only 1 phase)

simplest choice: $g(|A|^2) = \mu - \gamma |A|^2$

$$g(|A|^2)A = \underbrace{\mu A}_{\substack{\text{linear}\\ \text{term}\\ \text{(growth)}}} - \underbrace{(\gamma + iK)|A|^2 A}_{\substack{\text{lowest n.l. term} \Rightarrow \\ \text{(saturation)}}}$$

$\mu, \gamma > 0 \Rightarrow$ supercritical bifurcation

$\mu, \gamma < 0 \Rightarrow$ sub-critical bifurcation $\Rightarrow$ need h.o. to saturate.

Similarly, we have:

$$p(t) = \sum_n \left(p_n e^{in\omega t} + c.c.\right)$$

$$\int_0^{2\pi/\omega} d\tau \, \frac{e^{-i\omega t}}{i\omega} \epsilon p(t) = -i\epsilon E$$

so

$$\dot{A} = \underbrace{-i\frac{(\omega^2-\omega_0^2)}{2\omega}A}_{\text{mismatch}} + \underbrace{\mu A}_{\text{growth}} \underbrace{-(\gamma+iK)|A|^2A}_{\substack{\text{NL saturation} \\ \text{NL freq shift}}} \underbrace{-i\epsilon E}_{\text{drive}}$$

⇒ recovers CGL structure!

Note:

- derivation is "generic" to form of nonlinear oscillator.
- in general: $g(|A|^2) = \sum_n g_n (|A|^2)^n$

  with coeffs set by problem.

- not surprisingly, can also describe via method of reductive perturbation theory (i.e. Poincare-Lindstedt).

- in absence of forcing, recovers Landau-Stuart:

$$\frac{dA}{dt} = (1+i\eta)A - (1+i\alpha)|A|^2 A$$

- for validity, need:

$$\left.\begin{array}{l} |\omega-\omega_0| \ll \omega_0 \\ \mu \ll \omega_0 . \end{array}\right\} \text{ensure}$$

- NL term small
- weak instability of $A=0$ fixed point

Note: Story is consistent, i.e.

μA vs $\gamma |A|^2 A$ ensures:

so $|A|^2 \lesssim \mu/\gamma \Leftrightarrow NL \sim L$

$\Leftrightarrow$ requires small growth. In practice, CGL valid only near marginality.