Chapter 3.1

The Variational Principle for Problems of Ideal Magnetohydrodynamic Stability

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3.1.1. Introduction

In a variety of applications of plasma physics, for example controlled thermonuclear fusion, solar physics, and astrophysics, an important problem is the stability

Handbook of Plasma Physics, Eds. M.N. Rosenbluth and R.Z. Sagdeev
Volume 1: Basic Plasma Physics I, edited by A.A. Galeev and R.N. Sudan

against small perturbations of a steady state. Of particular significance is that class of relatively slow motions describable in fluid terms which, if they correspond to instability, lead to a radical change in the state, for example disruption of a pinch discharge. The rigorous theory of such motions, particularly when collisions are weak, leads to formidable mathematical problems. Thus one is led to analyze simpler idealized models of the phenomena, which are sufficiently tractable to lead to general results, and which qualitatively or semiquantitatively apply to the real situations.

The most useful such model is ideal magnetohydrodynamics. The plasma is viewed as a dissipation-free quasi-neutral fluid described in terms of its center of mass density, isotropic total pressure, and center of mass velocity, via equations representing conservation of mass, energy, and momentum. In this latter the force density due to the electric field acting on the charge density is neglected, as is the electromagnetic momentum. The electric field is eliminated from the problem by assuming perfect conductivity which allows one to relate the magnetic field strength directly to the kinematics of the center of mass fluid. A discussion of the physical significance of this description is to be found in Chapter 1.4. The resulting equations are the counterpart for systems in which magnetic forces are important of the Euler equations of ordinary fluid dynamics.

Even with the idealizations just described the linearized equations of motion governing the small departures from a desired steady state usually defy solution. Fortunately one is often not interested in the details of the motion, but rather in an answer to the simpler question of whether or not the system is unstable. That is, does there exist a class of initial conditions which within the framework of the linearized theory leads to exponential growth of the perturbations. The notion is that such unstable behavior will usually result in destruction of the desired steady configuration, even when limited by nonlinear effects. Clearly in order that these conclusions be significant it is necessary that the maximum growth rate characterizing the instability be greater than that characterizing the transport phenomena or weak flows present in the real situations being modeled. As will be shown, for the case of ideal magnetohydrodynamics the answer to this simpler question can be reduced to the determination of whether or not a functional W quadratic in the perturbed velocity can be made negative. This is analogous to the notion that a particle in a conservative field is in a stable equilibrium at the bottom of the potential well where all small displacements lead to an increase in potential energy, but unstable at a maximum of the potential energy where some small displacements lead to a decrease in potential energy.

The variational principle alluded to above and its generalizations form the basis of a vast body of work the results of which underly the present understanding of magnetohydrodynamic stability. The notion was first introduced by Lundquist (1951) and the general theory elaborated by Hain et al. (1957) and Bernstein et al. (1958). The demonstration that the results of the ideal magnetohydrodynamic variational principle yields a pessimistic result compared with a theory using the Vlasov equation was given by Kruskal and Oberman (1958) and completed by Grad (1966). A comprehensive bibliography is to be found in the volume by Bateman (1978).

This chapter will develop in detail the general theory for a plasma separated from perfectly conducting walls by a vacuum region, and apply it to two simple examples. An outline of the development is as follows: Section 3.1.2 defines the mathematical model, including the nonlinear equations of motion and the boundary conditions. The static equilibria employed and the linearized equations describing the small motions about them are presented in Section 3.1.3. Section 3.1.4 is devoted to a derivation of the variational principle, and Section 3.1.5 to the transformation and generalization of the resulting bilinear functional. Two applications are then given to demonstrate the techniques employed and the generality of the results obtained in relatively simple fashion. Section 3.1.6 derives a sufficient condition for stability for a magnetic field free plasma supported by a vacuum magnetic field in terms of the curvature of the bounding surface. Section 3.1.7 presents the derivation of Suydam's criterion for the stability of the diffuse linear pinch. Mathematical details are given in Appendices 3.1.A–3.1.D.

3.1.2. Basic equations of ideal magnetohydrodynamics

Consider a plasma where the total mass density ρ, center of mass velocity $\boldsymbol{v}$, total material stress tensor $\boldsymbol{P} = p\boldsymbol{I}$, electric field $\boldsymbol{E}$, and magnetic field $\boldsymbol{B}$ are governed by the equations of ideal magnetohydrodynamics, in Gaussian units,

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{1}$$

$$\rho(\partial\boldsymbol{v}/\partial t + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = -\nabla p + (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \tag{2}$$

$$(\partial/\partial t + \boldsymbol{v}\cdot\nabla)p/\rho^{\gamma} = 0 \tag{3}$$

$$\boldsymbol{E} + (1/c)\boldsymbol{v}\times\boldsymbol{B} = 0 \tag{4}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{5}$$

$$c\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t. \tag{6}$$

The current density associated with the description is $\boldsymbol{J} = (c/4\pi)\nabla\times\boldsymbol{B}$. Note that on taking its divergence (6) yields $(\partial/\partial t)\nabla\cdot\boldsymbol{B} = 0$, whence (5) is true if it holds initially. If (4) is used in (6) to eliminate $\boldsymbol{E}$, there results

$$\partial\boldsymbol{B}/\partial t = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\nabla\cdot\boldsymbol{v}. \tag{7}$$

In order to display certain features of the system which depend on the topology of the plasma and its surroundings, and to relate the theory to toroidal controlled fusion devices it will be assumed in developing the general theory that the plasma is a topological torus (region I) surrounded by an annular vacuum region II enclosed by a rigid perfectly conducting wall S′ as shown schematically in Figs. 3.1.1 and 3.1.2.

In the vacuum region it will be assumed that the time for a light signal to cross the system is much less than the time characterizing changes in $\boldsymbol{B}$, whence displacement current is negligible. Then since by assumption there is no conduction current

$$\nabla\times\boldsymbol{B} = 0 \tag{8}$$

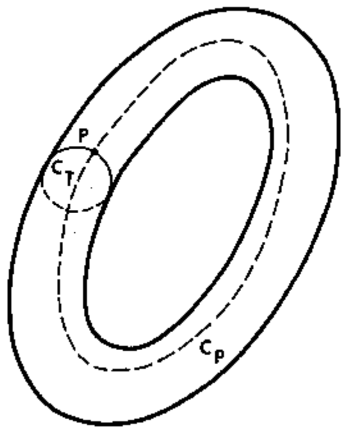

Fig. 3.1.1. Schematic view of a toroidal plasma indicating the paths of integration C_P and C_T.

which implies

$$\boldsymbol{B} = -\nabla\chi \tag{9}$$

whence following (5)

$$\nabla^2\chi = 0. \tag{10}$$

These equations must be supplemented by boundary conditions. For example on a rigid perfect conductor with unit surface normal $\boldsymbol{n}$, if $\boldsymbol{n}\cdot\boldsymbol{B} = 0$ initially, it must remain so, since for a fixed point on the immobile perfect conductor

$$\frac{\partial}{\partial t}(\boldsymbol{n}\cdot\boldsymbol{B}) = \boldsymbol{n}\cdot\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{n}\cdot\nabla\times\boldsymbol{E}. \tag{11}$$

But integrating over an arbitrary area in the wall and using Stokes's theorem gives

$$\oint \mathrm{d}^2 r\boldsymbol{n}\cdot\nabla\times\boldsymbol{E} = \oint \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{E} = 0 \tag{12}$$

since perfect conductivity implies that the tangential component of $\boldsymbol{E}$ vanishes on the wall. But, since the area of integration is arbitrary, (12) implies that at each point on the surface

$$\boldsymbol{n}\cdot\nabla\times\boldsymbol{E} = 0 \tag{13}$$

whence the right-hand side of (11) vanishes and $\boldsymbol{n}\cdot\boldsymbol{B}$ is a constant which is zero if so initially, as will be assumed.

A parallel situation prevails on the in general moving plasma–vacuum boundary by virtue of (4). To demonstrate this a convective derivative is denoted by a dot, whence, approaching the boundary from the plasma side,

$$\dot{\boldsymbol{B}} = \partial\boldsymbol{B}/\partial t + \boldsymbol{v}\cdot\nabla\boldsymbol{B} \tag{14}$$

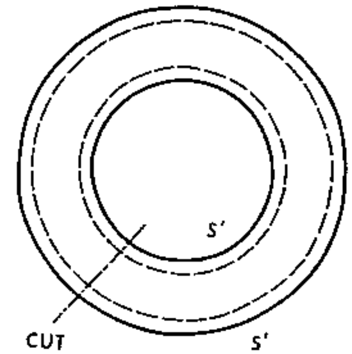

Fig. 3.1.2. Schematic view from above of a toroidal plasma surrounded by an annular vacuum region enclosed in rigid perfectly conducting walls.

etc. Then using (A7) for $\dot{\boldsymbol{n}}$, (7) and (14)

$$\begin{aligned}(\boldsymbol{n}\cdot\boldsymbol{B})^{\cdot} &= \dot{\boldsymbol{n}}\cdot\boldsymbol{B} + \boldsymbol{n}\cdot\dot{\boldsymbol{B}} \\ &= [-(\nabla\boldsymbol{v})\cdot\boldsymbol{n} + \boldsymbol{n}\boldsymbol{n}\cdot(\nabla\boldsymbol{v})\cdot\boldsymbol{n}]\cdot\boldsymbol{B} + \boldsymbol{n}\cdot[\boldsymbol{B}\cdot\nabla\boldsymbol{v} - \boldsymbol{B}\nabla\cdot\boldsymbol{v}] \\ &= \boldsymbol{n}\cdot\boldsymbol{B}\boldsymbol{n}\times(\boldsymbol{n}\times\nabla)\cdot\boldsymbol{v}.\end{aligned} \tag{15}$$

This is a first-order homogeneous ordinary differential equation for $\boldsymbol{n}\cdot\boldsymbol{B}$ along the trajectory of a fluid point on the boundary, the motion of which is governed by

$$\dot{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{r}, t). \tag{16}$$

Hence, as will be assumed, if $\boldsymbol{n}\cdot\boldsymbol{B}$ is initially zero, it stays zero. Moreover since it follows from $\nabla\cdot\boldsymbol{B} = 0$ that $\boldsymbol{n}\cdot\boldsymbol{B}$ is continuous at an interface, also $\boldsymbol{n}\cdot\boldsymbol{B} = 0$ on the vacuum side of S.

A dynamic boundary condition linking fluid and vacuum quantities can be obtained from (1)–(6) by noting that they imply conservation of momentum in the form

$$\frac{\partial}{\partial t}(\rho\boldsymbol{v}) + \nabla\cdot\left[\rho\boldsymbol{v}\boldsymbol{v} + \left(p + \frac{B^2}{8\pi}\right)\boldsymbol{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right] = 0. \tag{17}$$

The Cartesian components of (17) are all of the conservation form

$$\partial\sigma/\partial t + \nabla\cdot\boldsymbol{\Gamma} = 0 \tag{18}$$

where σ can be interpreted as a volume density and Γ as the associated flux density. Consider a set of nested surfaces with associated family of unit normals $\boldsymbol{n}$ as shown in Fig. 3.1.3. A surface of discontinuity S can be viewed as the limit as the thickness δ of a thin boundary layer goes to zero, such that σ varies rapidly in the direction of $\boldsymbol{n}$ but slowly perpendicular to $\boldsymbol{n}$ (see Figs. 3.1.3 and 3.1.4). It is convenient to write

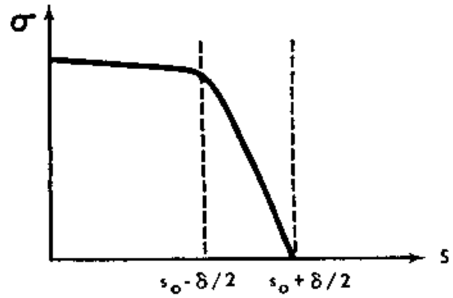

Fig. 3.1.3. Schematic plot of a density σ versus arc length s in a boundary layer of thickness δ.

(18) in the form

$$\boldsymbol{n}\cdot\nabla[\boldsymbol{n}\cdot(\boldsymbol{\Gamma}-\boldsymbol{v}\sigma)] = -\dot{\sigma} - \sigma\boldsymbol{n}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{v}) + \boldsymbol{n}\times(\boldsymbol{v}\times\boldsymbol{n})\cdot\nabla\sigma + \boldsymbol{n}\cdot(\nabla\boldsymbol{n})\cdot\boldsymbol{\Gamma} + \boldsymbol{n}\times(\boldsymbol{n}\times\nabla)\cdot\boldsymbol{\Gamma} \tag{19}$$

where $\dot{\sigma} = \partial\sigma/\partial t + \boldsymbol{v}\cdot\nabla\sigma$ is the time derivative of σ as seen by an observer moving with the fluid. Note that all the terms on the right-hand side of (19) do not vary rapidly as one traverses the boundary layer along a curve everywhere parallel to $\boldsymbol{n}$. Thus if s denotes arc length, one writes $\boldsymbol{n}\cdot\nabla = \partial/\partial s$, and integrates (19) across the boundary layer, obtaining in the limit $\delta \to 0$

$$\langle \boldsymbol{n}\cdot(\boldsymbol{\Gamma}-\boldsymbol{v}\sigma)\rangle = 0. \tag{20}$$

In (20) the notation $\langle\ \rangle$ denotes the jump in the enclosed quantity. Equation (20), when applied to (17), yields, since $\boldsymbol{n}\cdot\boldsymbol{B} = 0$,

$$\langle p + B^2/8\pi\rangle = 0. \tag{21}$$

Equation (21) is a statement of balance of total pressure, material plus magnetic, required in order that in a region of rapid variation of density, pressure, etc., there not be infinite acceleration of an element of mass.

In the annular vacuum region II, in addition to the boundary condition

$$\boldsymbol{n}\cdot\boldsymbol{B} = 0 \tag{22}$$

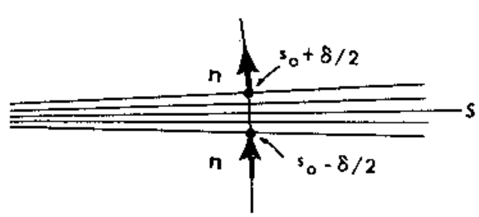

Fig. 3.1.4. Schematic diagram displaying nested surfaces on which a density σ is constant and indicating the path of integration tangent to the slowly varying unit normal $\boldsymbol{n}$.

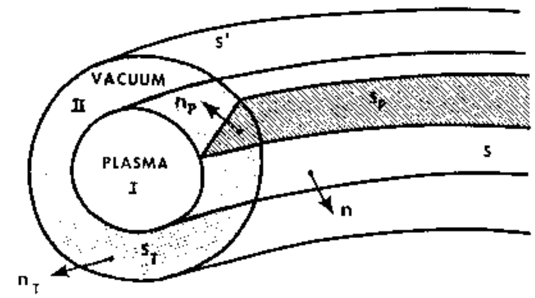

Fig. 3.1.5. Schematic diagram of cut toroidal regions showing washer-like surface of integration S_T (stippled), and ribbon-like surface S_P (shaded).

which will be assumed to prevail on S and S′, it is necessary in order for a nontrivial solution, given the boundary condition $\boldsymbol{n}\cdot\boldsymbol{B} = 0$, to stipulate the toroidal flux

$$\Phi_T = \int_{S_T} d^2r\,\boldsymbol{n}_T\cdot\boldsymbol{B} = -\int d^2r\,\boldsymbol{n}_T\cdot\nabla\chi \tag{23}$$

and the poloidal flux

$$\Phi_P = \int_{S_P} d^2r\,\boldsymbol{n}_P\cdot\boldsymbol{B} = -\int d^2r\,\boldsymbol{n}_P\cdot\nabla\chi \tag{24}$$

where S_P is a washer-like surface such as is shown on Fig. 3.1.5 in a section cutting the torus the short way, and S_P is a similar ribbon in a section slicing the torus the long way. These fluxes are independent of time, for, using (A11),

$$\begin{aligned}\frac{d\Phi_T}{dt} &= \int_{S_T} d^2\boldsymbol{r}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \int_{C_T} d\boldsymbol{r}\cdot\boldsymbol{v}\times\boldsymbol{B}\\ &= -\int d^2\boldsymbol{r}\cdot c\nabla\times\boldsymbol{E} - \int d\boldsymbol{r}\cdot\boldsymbol{v}\times\boldsymbol{B}\\ &= -c\int_{C'_T} d\boldsymbol{r}\cdot\boldsymbol{E} - c\int_{C_T} d\boldsymbol{r}\cdot\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right).\end{aligned} \tag{25}$$

But, on C'_T, $\boldsymbol{n}\times\boldsymbol{E} = 0$, while on S it follows from (6) that

$$\langle \boldsymbol{n}\times[\boldsymbol{E} + (1/c)\boldsymbol{v}\times\boldsymbol{B}]\rangle = 0 \tag{26}$$

whence, since (7) holds on the plasma side, it follows that $\boldsymbol{n}\times[\boldsymbol{E} + (1/c)\boldsymbol{v}\times\boldsymbol{b}] = 0$ on the vacuum side. Thus the right-hand side of (25) vanishes. A similar proof holds for Φ_p. The fluxes are also independent of the choice of cut provided it yields a surface of the same topological character. Thus if S'_T is obtained by a different sectioning of the torus as shown in Fig. 3.1.6, if one integrates $\nabla\cdot\boldsymbol{B} = 0$ over the volume interior to S, S′, S_P, and S'_P and applies Gauss's theorem, the contributions from those parts of the boundary lying in S and S′ vanish because there $\boldsymbol{n}\cdot\boldsymbol{B} = 0$,

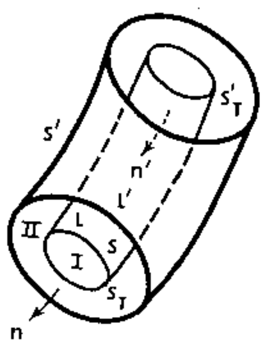

Fig. 3.1.6. Schematic diagram of a segment of toroidal domain.

whence the conclusion follows on taking account of the direction of the unit normals to S_T and S'_T.

3.1.3. *Linearized description about a static equilibrium*

We shall be concerned with the stability of static equilibria, that is solutions of the governing equations for which the velocity vanishes and which are time independent. Then (1), (3), and (6) are satisfied automatically since it follows from (4) that the electric field in the plasma vanishes, whence (6) is satisfied. If quantities associated with this state are distinguished by a subscript zero, (2) and (5) become

$$\nabla p_0 = (1/4\pi)(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_0 \tag{27}$$

$$\nabla \cdot \boldsymbol{B}_0 = 0. \tag{28}$$

In the vacuum region, distinguishing the vacuum magnetic field by a circumflex,

$$\hat{\boldsymbol{B}}_0 = -\nabla \chi_0 \tag{29}$$

$$\nabla^2 \chi_0 = 0 \tag{30}$$

On the static interface S between the plasma and the vacuum

$$p_0 + \frac{B_0^2}{8\pi} - \frac{\hat{B}_0^2}{8\pi} = 0 \tag{31}$$

$$\boldsymbol{n}_0 \cdot \boldsymbol{B}_0 = 0. \tag{32}$$

$$\boldsymbol{n}_0 \cdot \hat{\boldsymbol{B}}_0 = 0 \tag{33}$$

and on the rigid perfect conductor S′

$$\boldsymbol{n} \cdot \hat{\boldsymbol{B}}_0 = 0. \tag{34}$$

The two fluxes

$$\Phi_{P0} = \int_{S_{P0}} d^2 r \boldsymbol{n}_{P0} \hat{\boldsymbol{B}}_0 \tag{35}$$

$$\Phi_{T0} = \int_{S_{T0}} d^2 r \boldsymbol{n}_{T0} \cdot \hat{\boldsymbol{B}}_0 \tag{36}$$

are assumed to be given. This system of equations has been shown to have solutions for the case of axisymmetric toroidal symmetry. The general theory, however, is not completely understood, but conventionally one assumes that a solution exists and examines the consequences as regards stability as follows.

Consider a small motion about a static equilibrium such as defined above and write $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$, $\boldsymbol{v} = 0 + \boldsymbol{v}_1$, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$, $\boldsymbol{E} = 0 + \boldsymbol{E}_1$, etc., where $|\rho_1| \ll \rho_0$, $|p_1| \ll p_0$, etc. If (1), (2), (3), (5), and (7) are linearized there results

$$\partial \rho_1/\partial t + \nabla \cdot (\rho_0 \boldsymbol{v}_1) = 0 \tag{37}$$

$$\rho_0 \partial \boldsymbol{v}_1/\partial t = -\nabla p_1 + (1/4\pi)(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 + (1/4\pi)(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_1 \tag{38}$$

$$\partial p_1/\partial t + \boldsymbol{v}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{v}_1 = 0 \tag{39}$$

$$\nabla \cdot \boldsymbol{B}_1 = 0 \tag{40}$$

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_1). \tag{41}$$

In writing (43) the exact equation

$$\partial p/\partial t + \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} = 0 \tag{42}$$

has been used which was obtained using (1) to eliminate ρ from (3). If one takes the time derivative of (38), and uses (37), (39), and (41) to eliminate $\partial \rho_1/\partial t$, $\partial p_1/\partial t$, and $\partial \boldsymbol{B}_1/\partial t$ there results

$$\rho_0 \partial^2 \boldsymbol{v}_1/\partial t^2 = \boldsymbol{F} \boldsymbol{v}_1 \tag{43}$$

where the linear operator $\boldsymbol{F}$ acting on a vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{F}\boldsymbol{\xi} = \nabla(\gamma p_0 \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p_0) + (1/4\pi)(\nabla \times \boldsymbol{Q}) \times \boldsymbol{B}_0 + (1/4\pi)(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{Q} \tag{44}$$

and

$$\boldsymbol{Q} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0). \tag{45}$$

In the vacuum region, if where necessary quantities are distinguished by a circumflex, one has

$$\nabla \cdot \hat{\boldsymbol{B}}_1 = 0 \rightarrow \nabla \cdot \partial \hat{\boldsymbol{B}}_1/\partial t = 0 \tag{46}$$

$$\nabla \times \hat{\boldsymbol{B}}_1 = 0 \rightarrow \nabla \times \partial \hat{\boldsymbol{B}}_1/\partial t = 0. \tag{47}$$

Clearly following (46) one can write

$$\partial \boldsymbol{B}_1/\partial t = \nabla \times \boldsymbol{A}_1 \tag{48}$$

where $-c\boldsymbol{A}_1$ is the perturbed electric field, and (47) implies

$$\nabla \times \nabla \times \boldsymbol{A}_1 = 0. \tag{49}$$

Alternatively it follows from (47) that

$$\partial \hat{\boldsymbol{B}}_1/\partial t = -\nabla \chi_1 \tag{50}$$

whence (46) requires

$$\nabla^2 \chi_1 = 0. \tag{51}$$

One is at liberty to use either the vector potential $\boldsymbol{A}_1$ or the scalar potential χ_1. The former is more convenient in derivations, the latter for use with the energy principle to be derived. On the rigid perfectly conducting wall S′ the condition $\boldsymbol{n}\cdot\boldsymbol{B}=0$ yields on linearization, since $\boldsymbol{n}$ there does not change in time,

$$0 = \boldsymbol{n}_0\cdot\partial\hat{\boldsymbol{B}}_1/\partial t = \boldsymbol{n}_0\cdot\nabla\times\boldsymbol{A}_1 = -\boldsymbol{n}_0\cdot\nabla\chi_1. \tag{52}$$

Note that for any area in S′ (52) implies

$$0 = \int d^2\boldsymbol{r}\cdot\nabla\times\boldsymbol{A}_1 = \oint d\boldsymbol{r}\cdot\boldsymbol{A}_1. \tag{53}$$

Since the closed line over which the integral in (53) is performed is arbitrary, it follows that

$$\boldsymbol{n}_0\times\boldsymbol{A}_1 = \boldsymbol{n}\times(\boldsymbol{n}\times\nabla)\lambda \tag{54}$$

where λ is a scalar defined on S′ such that $\boldsymbol{n}\times\nabla\lambda$ is single valued. We shall choose the gauge such that $\lambda=0$. On the moving boundary S, the time derivative as seen by an observer moving with the fluid of $\boldsymbol{n}\cdot\boldsymbol{B}=0$ yields, as for (15),

$$\begin{aligned}0 &= \dot{\boldsymbol{n}}\cdot\boldsymbol{B}+\boldsymbol{n}\cdot\dot{\boldsymbol{B}}\\ &= [-(\nabla\boldsymbol{v})\cdot\boldsymbol{n}+\boldsymbol{n}\boldsymbol{n}\cdot(\nabla\boldsymbol{v})\cdot\boldsymbol{n}]\cdot\hat{\boldsymbol{B}}+\boldsymbol{n}\cdot[\partial\hat{\boldsymbol{B}}/\partial t+\boldsymbol{v}\cdot\nabla\hat{\boldsymbol{B}}]\\ &= \boldsymbol{n}\cdot[\partial\hat{\boldsymbol{B}}/\partial t+\boldsymbol{v}\cdot\nabla\hat{\boldsymbol{B}}-\hat{\boldsymbol{B}}\cdot\nabla\boldsymbol{v}+\boldsymbol{B}\,\nabla\cdot\boldsymbol{v}]\\ &= \boldsymbol{n}\cdot[\partial\hat{\boldsymbol{B}}/\partial t-\nabla\times(\boldsymbol{v}\times\hat{\boldsymbol{B}})].\end{aligned} \tag{55}$$

On linearization (54) yields

$$0 = \boldsymbol{n}_0\cdot[\partial\hat{\boldsymbol{B}}_1/\partial t-\nabla\times(\boldsymbol{v}_1\times\hat{\boldsymbol{B}}_0)] \tag{56}$$

or, using (48), (55) requires

$$0 = \boldsymbol{n}_0\cdot\nabla\times(\boldsymbol{A}_1-\boldsymbol{v}_1\times\hat{\boldsymbol{B}}_0) = 0 \tag{57}$$

which with a suitable choice of gauge implies, since $\boldsymbol{n}_0\cdot\hat{\boldsymbol{B}}_0=0$,

$$0 = \boldsymbol{n}_0\times(\boldsymbol{A}_1-\boldsymbol{v}_1\times\boldsymbol{B}_0) = \boldsymbol{n}\times\boldsymbol{A}_1+\boldsymbol{n}_0\cdot\boldsymbol{v}_1\hat{\boldsymbol{B}}_0. \tag{58}$$

When (50) is employed (55) requires

$$0 = \boldsymbol{n}_0\cdot[\nabla\chi_1+\nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)]. \tag{59}$$

Note that on integrating over an arbitrary element of surface in S_0, since $\boldsymbol{B}_0\times d\boldsymbol{r}$ is

parallel to $\boldsymbol{n}_0$

$$\begin{aligned}\int d^2r\,\boldsymbol{n}_0\cdot\nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0) &= \oint d\boldsymbol{r}\cdot\boldsymbol{v}_1\times\boldsymbol{B}_0\\ &= \oint\boldsymbol{v}_1\cdot\boldsymbol{B}_0\times d\boldsymbol{r}\\ &= \oint\boldsymbol{n}_0\cdot\boldsymbol{v}_1\boldsymbol{n}_0\cdot\boldsymbol{B}_0\times d\boldsymbol{r}\\ &= \oint d\boldsymbol{r}\cdot\boldsymbol{n}_0\times\boldsymbol{B}_0\boldsymbol{n}_0\cdot\boldsymbol{v}_1\\ &= \int d^2r\,\boldsymbol{n}_0\cdot\nabla\times(\boldsymbol{n}_0\times\boldsymbol{B}_0\boldsymbol{n}_0\cdot\boldsymbol{v}_1).\end{aligned} \tag{60}$$

Since the area of integration is arbitrary it follows that

$$\begin{aligned}\boldsymbol{n}_0\cdot\nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0) &= \boldsymbol{n}_0\cdot\nabla\times(\boldsymbol{n}_0\times\boldsymbol{B}_0\boldsymbol{n}_0\cdot\boldsymbol{v}_1)\\ &= (\boldsymbol{n}_0\times\nabla)\cdot(\boldsymbol{n}_0\times\boldsymbol{B}_0\boldsymbol{n}_0\cdot\boldsymbol{v}_1)\end{aligned} \tag{61}$$

which involves only $\boldsymbol{n}_0\cdot\boldsymbol{v}$, and $\boldsymbol{n}_0\times\nabla$, that is only derivatives tangent to S_0. The flux condition (25) yields on linearization

$$0 = \int_{S_{T0}} d^2\boldsymbol{r}\cdot\frac{\partial\hat{\boldsymbol{B}}_1}{\partial t} - \int_{C_{T0}} d\boldsymbol{r}\cdot\boldsymbol{v}_1\times\hat{\boldsymbol{B}}_0. \tag{62}$$

When (48) is employed with boundary conditions (55) and (57), (62) is satisfied automatically, as is immediately seen on applying Stokes' theorem. When (50) is employed, (62) implies

$$0 = \int_{S_{T0}} d^2\boldsymbol{r}\cdot\nabla\chi_1 + \int_{C_{T0}} d\boldsymbol{r}\cdot\boldsymbol{v}_1\times\hat{\boldsymbol{B}}_0. \tag{63}$$

Parallel conclusions hold as regards the poloidal flux, where

$$0 = \int_{S_{P0}} d^2\boldsymbol{r}\cdot\nabla\chi_1 + \int_{C_{P0}} d\boldsymbol{r}\cdot\boldsymbol{v}_1\times\hat{\boldsymbol{B}}_0. \tag{64}$$

The connective time derivative of (21) yields

$$\frac{\partial p}{\partial t}+\boldsymbol{v}\cdot\nabla p+\frac{1}{4\pi}\boldsymbol{B}\cdot\left(\frac{\partial\boldsymbol{B}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{B}\right) = \frac{1}{4\pi}\hat{\boldsymbol{B}}\cdot\left(\frac{\partial\hat{\boldsymbol{B}}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{B}\right) \tag{65}$$

which on linearization requires, on using (39), (41), and (48),

$$\begin{aligned}&-\gamma p_0\nabla\cdot\boldsymbol{v}+(1/4\pi)\boldsymbol{B}_0\cdot[\nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)+\boldsymbol{v}_1\cdot\nabla\boldsymbol{B}]\\ &\quad= (1/4\pi)\boldsymbol{B}_0\cdot[\nabla\times\boldsymbol{A}_1+\boldsymbol{v}_1\cdot\nabla\boldsymbol{B}].\end{aligned} \tag{66}$$

Henceforth, for notational simplicity, the subscripts zero and one will be suppressed.

3.1.4. The variational principle

It will now be shown that (43) implies a conservation law. To this end the scalar product of (43) with $\partial v/\partial t$ is formed and integrated over the volume of the plasma. The property shown in Appendix C that F is self-adjoint will be used, namely that on integrating over the plasma region

$$\int d^3v\, v\cdot Fu = \int d^3r\, u\cdot Fv. \tag{67}$$

Then if the quadratic form (the kinetic energy)

$$K = \int d^3r \tfrac{1}{2}\rho(\partial v/\partial t)^2 \tag{68}$$

one has, since the domain of integration is time-dependent, and $\partial/\partial t(\partial v/\partial t)^2 = \partial v/\partial t\cdot\partial^2 v/\partial t^2$, on employing (43),

$$\begin{aligned}\frac{dK}{dt} &= \int_1 d^3r\,\frac{\partial v}{\partial t}\cdot Fv\\ &= \frac{1}{2}\int d^3r\left(\frac{\partial v}{\partial t}\cdot Fv + v\cdot F\frac{\partial v}{\partial t}\right)\\ &= -dW/dt,\end{aligned} \tag{69}$$

where

$$W(v) = -\frac{1}{2}\int_1 d^3r\, v\cdot Fv. \tag{70}$$

Thus

$$K + W = \text{constant} = E. \tag{71}$$

Since (43) is a linear equation with time-independent coefficients, one can seek eigensolutions of the form

$$v(r,t) = e^{i\omega t}\xi(r) \tag{72}$$

which when inserted in (43) yields

$$-\omega^2\rho\xi = F\xi. \tag{73}$$

It is readily seen from the self-adjointness property (67) that the eigenvalues ω^2 are real. In general, however, the spectrum is very complex, consisting of both point and continuous eigenvalues, while the eigenvalue $\omega^2 = 0$ is infinitely degenerate, since the choice $\xi = \alpha(p)B + \beta(p)\nabla\times B$ with α and β arbitrary functions of p makes $F\xi = 0$. Moreover the task of solving (73) is insuperable except for the cases of slab or cylindrical symmetry, and very simple magnetic field configurations and pressure profiles. Fortunately for many applications all that is desired is a yes–no answer to the question of stability. As will be seen, on choosing a physically plausible definition of stability, the problem can be reduced to ascertaining whether or not the functional W of (70) can be made negative.

Now the linear growth in time associated with $\omega^2 = 0$ is physically not significant since in practice the ideal magnetohydrodynamic model is but an approximation to real systems involving weak flows usually due to transport associated with collisions or turbulence. A dangerous instability is one which grows in a time short compared with τ, that characterizing the weak flows. As will be seen, apart from $\omega^2 = 0$, any unstable motions grow exponentially in time and may be expected to be dangerous if the effective growth rate is greater than $1/\tau$.

Now it is evident from (71) that if $W > 0$ then the non-negative functional K cannot grow without bound in time, since (71) implies

$$K = E - W \leqslant E = \text{constant}. \tag{74}$$

Thus it is sufficient for stability defined as boundedness of K that W be non-negative.

It will now be shown that when W can be made negative the positive functional, the kinetic energy

$$I = \frac{1}{2}\int d^3r\rho v^2 \tag{75}$$

will grow exponentially in time, which will be deemed *instability*. To this end note that, denoting a time derivative by a dot,

$$\dot I = \int d^3r\rho v\cdot\frac{\partial v}{\partial t} \tag{76}$$

and on using (43), (68), and (70)

$$\ddot I = \int d^3r\rho\left(\frac{\partial v}{\partial t}\right)^2 + \int d^3r\rho v\cdot\frac{\partial^2 v}{\partial t^2} = 2K + \int d^3r\, v\cdot Fv = 2K - 2W. \tag{77}$$

Thus

$$\begin{aligned}(\ln I)^{\cdot\cdot} &= \frac{\ddot I}{I} - \frac{\dot I^2}{I^2}\\ &= \frac{1}{I}\left[-2W + 2K - \frac{1}{I}\left(\int d^3r\rho v\cdot\frac{\partial v}{\partial t}\right)^2\right]\\ &\geqslant \frac{1}{I}[-2W + 2K - 4K] = -\frac{2E}{I}\end{aligned} \tag{78}$$

since by Schwartz's inequality

$$\left(\int d^3r\rho v\cdot\frac{\partial v}{\partial t}\right)^2 \leqslant \left(\int d^3r\rho v^2\right)\left(\int d^3r\rho\left(\frac{\partial v}{\partial t}\right)^2\right) = 4IK. \tag{79}$$

Suppose there is a velocity field $v_0(r)$ which satisfies all the various boundary conditions and which makes $W < 0$. Consider a motion such that initially $v = v_0$ and $\partial v/\partial t = 0$. Then the associated value of E is negative. Let $y = \ln I(t)/I(0)$ and define $E = -2\nu^2 I(0)$. Then (78) implies

$$\ddot y \geqslant 2\nu^2 e^{-y} \tag{80}$$

and $y(0)=0$, $\dot{y}(0)=0$. Let $Y(t)$ satisfy these same initial conditions and (80) with the equality sign. Clearly $y \geqslant Y$. But a first integral can be obtained by multiplying $\ddot{Y} = 2\nu^2 e^{-Y}$ by $\dot{Y}$, namely

$$\tfrac{1}{2}\dot{Y}^2 = 2\nu^2(1-e^{-Y}) \tag{81}$$

while a second integration gives

$$2\nu t = \int_0^Y dy\,(1-e^{-y})^{-1/2} = \ln\frac{1+(1-e^{-Y})}{1-(1-e^{-Y})^{1/2}}. \tag{82}$$

Thus

$$Y = \ln(\cosh \nu t)^2 \tag{83}$$

and

$$I(t) \geqslant I(0)\,\tfrac{1}{4}(e^{2\nu t}+2+e^{-2\nu t}). \tag{84}$$

Clearly $I(t)$ grows exponentially in time. Thus with the definitions here adopted the necessary and sufficient condition for stability is that W be non-negative.

3.1.5. Transformation and generalization of the potential energy W

In order to investigate W, it is convenient to transform (70) into a more convenient form. That is one writes on using (44) and (70)

$$\begin{aligned} W(\boldsymbol{\xi}) &= -\frac{1}{2}\int_{\mathrm{I}} d^3r\,\boldsymbol{\xi}\cdot \boldsymbol{F}\boldsymbol{\xi} \\ &= -\frac{1}{2}\int_{\mathrm{I}} d^3r\,\boldsymbol{\xi}\cdot\Big(\nabla(\gamma p\,\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p) \\ &\qquad + \frac{1}{4\pi}(\nabla\times\boldsymbol{Q})\times\boldsymbol{B} + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}\Big) \\ &= -\frac{1}{2}\int_{\mathrm{I}} d^3r\Big\{\nabla\cdot\Big(\boldsymbol{\xi}(\gamma p\,\nabla\cdot\boldsymbol{\xi}+\boldsymbol{\xi}\cdot\nabla p) - \frac{1}{4\pi}\boldsymbol{Q}\times(\boldsymbol{\xi}\times\boldsymbol{B})\Big) \\ &\qquad -(\nabla\cdot\boldsymbol{\xi})(\gamma p\,\nabla\cdot\boldsymbol{\xi}+\boldsymbol{\xi}\cdot\nabla p) - \frac{1}{4\pi}Q^2 + \frac{1}{4\pi}\boldsymbol{\xi}\cdot(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}\Big\}. \end{aligned} \tag{85}$$

Define

$$W_{\mathrm{F}}(\boldsymbol{\xi}) = \frac{1}{2}\int_{\mathrm{I}} d^3r\Big(\gamma p(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi}\cdot\nabla p + \frac{1}{4\pi}Q^2 + \frac{1}{4\pi}\boldsymbol{\xi}\times\boldsymbol{Q}\cdot\nabla\times\boldsymbol{B}\Big). \tag{86}$$

Then on using Gauss's theorem since $\boldsymbol{n}\cdot\boldsymbol{B}=0$, one can express (85) as,

$$W - W_{\mathrm{F}} = -\frac{1}{2}\int_{\mathrm{S}} d^2r\,\boldsymbol{n}\cdot\boldsymbol{\xi}\Big[\gamma p\,\nabla\cdot\boldsymbol{\xi} - \frac{1}{4\pi}\boldsymbol{B}\cdot(\boldsymbol{Q}+\boldsymbol{\xi}\cdot\nabla\boldsymbol{B}) + \boldsymbol{\xi}\cdot\nabla\Big(p+\frac{B^2}{8\pi}\Big)\Big], \tag{87}$$

where the unit normal $\boldsymbol{n}$ is directed outward from the plasma. On using boundary condition (66) to eliminate $\gamma p\,\nabla\cdot\boldsymbol{\xi}$ from (87) one obtains

$$W - W_{\mathrm{F}} = -\frac{1}{2}\int d^2r\,\boldsymbol{n}\cdot\boldsymbol{\xi}\Big[-\frac{1}{4\pi}\hat{\boldsymbol{B}}\cdot\nabla\times\boldsymbol{A} + \boldsymbol{\xi}\cdot\nabla\Big(p+\frac{B^2}{8\pi}-\frac{\hat{B}^2}{8\pi}\Big)\Big]. \tag{88}$$

But everywhere on S, $p + B^2/8\pi - \hat{B}^2/8\pi = 0$, whence $\boldsymbol{n}\times\nabla(p+B^2/8\pi-\hat{B}^2/8\pi) = 0$. Thus if one defines

$$W_{\mathrm{S}} = -\frac{1}{2}\int d^2r\,(\boldsymbol{n}\cdot\boldsymbol{\xi})^2\boldsymbol{n}\cdot\nabla\Big(p+\frac{B^2}{8\pi}-\frac{\hat{B}^2}{8\pi}\Big) \tag{89}$$

and uses boundary conditions (54) and (58), then (88) leads to

$$\begin{aligned} W - W_{\mathrm{F}} - W_{\mathrm{S}} &= -\frac{1}{8\pi}\int_{\mathrm{S}} d^2r\,\boldsymbol{n}\times\boldsymbol{A}\cdot\nabla\times\boldsymbol{A} \\ &= \frac{1}{8\pi}\int_{\mathrm{S+S'}} d^2r(-\boldsymbol{n})\cdot\boldsymbol{A}\times(\nabla\times\boldsymbol{A}) \\ &= \frac{1}{8\pi}\int_{\mathrm{II}} d^3r\,\nabla\cdot[\boldsymbol{A}\times(\nabla\times\boldsymbol{A})] \\ &= \frac{1}{8\pi}\int_{\mathrm{II}} d^3r(\nabla\times\boldsymbol{A})^2. \end{aligned} \tag{90}$$

In transforming (90) the fact has been used that $-\boldsymbol{n}$ is the outward normal to S as seen from the vacuum side, and the integral over S′ is zero in virtue of (54) with $\lambda = 0$. Thus if one introduces

$$W_{\mathrm{V}} = \frac{1}{8\pi}\int_{\mathrm{II}} d^3r(\nabla\times\boldsymbol{A})^2 \tag{91}$$

then

$$W = W_{\mathrm{F}} + W_{\mathrm{S}} + W_{\mathrm{V}}. \tag{92}$$

We are concerned with whether W can be made negative. A systematic way of investigating this question is to seek to minimize W with respect to $\boldsymbol{\xi}$ subject to any suitable norm which keeps W bounded. Since $\nabla\times\nabla\times\boldsymbol{A}=0$ is the Euler–Lagrange equation determining that vector potential which minimizes the non-negative functional W_{V} subject to $\boldsymbol{n}\times\boldsymbol{A} = -\boldsymbol{n}\cdot\boldsymbol{\xi}\hat{\boldsymbol{B}}$ on S and $\boldsymbol{n}\times\boldsymbol{A}=0$ on S′, it is sufficient to minimize W with respect to $\boldsymbol{A}$, rather than require that $\nabla\times\nabla\times\boldsymbol{A}=0$. Moreover, in the minimization one need not require that the pressure balance condition (66) be satisfied, for it will be shown that if there are any vector fields $\bar{\boldsymbol{\xi}}$ and $\boldsymbol{A}$ which make W negative but do not satisfy (66) one can always find neighboring fields $\boldsymbol{\xi}$ and $\boldsymbol{A}$ which also make W negative but do satisfy (66). To demonstrate this let $\boldsymbol{\xi} = \bar{\boldsymbol{\xi}} + \varepsilon\boldsymbol{\eta}$, where $\boldsymbol{\eta}$, which is of order $\bar{\boldsymbol{\xi}}$, changes rapidly in an arbitrarily small distance of order ε as one moves normally from S into the fluid, but has slowly varying derivatives in directions tangent to S in this boundary layer, and $\nabla\boldsymbol{\eta}$ is or order $\nabla\bar{\boldsymbol{\xi}}$ outside the boundary layer. Then

$$\nabla\cdot(\varepsilon\eta) = [\boldsymbol{nn}\cdot\nabla - \boldsymbol{n}\times(\boldsymbol{n}\times\nabla)]\cdot(\varepsilon\eta)$$
$$= \boldsymbol{n}\cdot\nabla(\varepsilon\boldsymbol{n}\cdot\eta) - \boldsymbol{n}\cdot(\nabla\boldsymbol{n})\cdot\varepsilon\eta - \boldsymbol{n}\times(\boldsymbol{n}\times\nabla)\cdot(\varepsilon\eta)$$
$$= \mathrm{O}(\boldsymbol{n}\cdot\eta) \tag{93}$$

since $\boldsymbol{n}\cdot(\nabla\boldsymbol{n})$ is of order unity, as are tangential derivatives $\boldsymbol{n}\times\nabla$. But the pressure balance condition (66) can be written

$$-\left(\gamma p + \frac{B^2}{8\pi}\right)\nabla\cdot\xi + \frac{1}{4\pi}\boldsymbol{B}\cdot(\nabla\xi)\cdot\boldsymbol{B} = \frac{1}{4\pi}\hat{\boldsymbol{B}}\cdot[\nabla\times\boldsymbol{A} + \xi\cdot\nabla\hat{\boldsymbol{B}}] \tag{94}$$

whence correct to zeroth order in ε, since $\boldsymbol{B}\cdot\nabla$ involves no normal derivatives,

$$-\left(\gamma p + \frac{B^2}{8\pi}\right)[\nabla\cdot\tilde{\xi} + \varepsilon\boldsymbol{n}\cdot\nabla(\boldsymbol{n}\cdot\eta)] + \frac{1}{4\pi}\boldsymbol{B}\cdot(\nabla\tilde{\xi})\cdot\boldsymbol{B}$$
$$= \frac{1}{4\pi}\hat{\boldsymbol{B}}\cdot[\nabla\times\boldsymbol{A} + \tilde{\xi}\cdot\nabla\hat{\boldsymbol{B}}] \tag{95}$$

which determines $\boldsymbol{n}\cdot\nabla(\boldsymbol{n}\cdot\eta)$ on S. Since (95) involves only the normal derivative of $\boldsymbol{n}\cdot\eta$ and not $\boldsymbol{n}\cdot\eta$ itself, $\boldsymbol{n}\cdot\eta = 0$ can be chosen on S. Thus W_S and W_V need not be changed on using ξ in place of $\tilde{\xi}$. Note, moreover, that the only changes of order unity in the integrand of W_F resulting from using ξ in place of $\tilde{\xi}$ occur via those terms involving $\nabla\cdot\xi$, in the boundary layer of thickness ε near S. Outside the boundary layer the changes are of order ε. Thus the associated change in W_F is of order ε, and can be made arbitrarily small. Consequently one can dispense with (66) in minimizing W.

One can, if desired, so as to deal with a scalar rather than a vector, use χ in place of $\boldsymbol{A}$. It is only necessary to write

$$W_V = \frac{1}{8\pi}\int_{II} d^3r(\nabla\chi)^2 \tag{96}$$

and employ the boundary conditions (52) and (59). The constraints (63) and (64) must also be included in order to have a unique minimum.

The program of determining whether W can be made negative is conveniently carried out as follows. Consider first the class of ξ for which $\boldsymbol{n}\cdot\xi = 0$ on S. Then $W_S = 0$ and $W_V = 0$. Using (D13)

$$8\pi W_F = \int_I d^3r\{4\pi\gamma p(\nabla\cdot\xi)^2 + [\nabla\times(\xi\times\boldsymbol{B}) + (\nabla\times\boldsymbol{B})\times\boldsymbol{n}(\boldsymbol{n}\cdot\xi)]^2$$
$$-2(\boldsymbol{n}\cdot\xi)^2(\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{n})\}. \tag{97}$$

Clearly in order to bound W_F from below it is only necessary to norm $\boldsymbol{n}\cdot\xi$, for example to require

$$\int_I d^3r(\boldsymbol{n}\cdot\xi)^2 = 1.$$

But, as shown in Appendix 3.1.D, if one writes $\xi = a\nabla p + b\nabla\times\boldsymbol{B} + c\boldsymbol{B}$, then

$$8\pi W_F = \int_I d^3r\{[\nabla\times(-a\boldsymbol{B}\times\nabla p) + 4\pi(\nabla b)\times\nabla p + a(\nabla\times\boldsymbol{B})\times\nabla p]^2$$
$$+4\pi\gamma p[\nabla\cdot(a\nabla p) + (\nabla b)\cdot\nabla\times\boldsymbol{B} + \boldsymbol{B}\cdot\nabla c]^2$$
$$-2a^2(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot(\boldsymbol{B}\cdot\nabla\nabla p)\}. \tag{98}$$

Note that the integrand in (98) involves no derivatives of b and c in the direction of ∇p. Thus the Euler equations resulting from seeking to minimize W_F with respect to b and c holding a fixed will not link the values of a and b on different magnetic surfaces p = constant, and are hence partial differential equations in two rather than three variables. Finally the potentially negative term in the the integrand of (98) can be written in terms of the differential geometry of the lines of force defined by the system of ordinary differential equations $d\boldsymbol{r}/d\lambda = \boldsymbol{B}$. Then $\boldsymbol{t} = \boldsymbol{B}/B$ is the unit tangent, and if $\boldsymbol{\nu}$ is the unit normal and $\boldsymbol{b}$ the binormal, these unit vectors obey the Serret–Frenet formulas

$$\boldsymbol{t}\cdot\nabla\boldsymbol{b} = \tau\boldsymbol{\nu}, \qquad \boldsymbol{t}\cdot\nabla\boldsymbol{\nu} = \tau\boldsymbol{b} - \kappa\boldsymbol{t}, \qquad \boldsymbol{t}\cdot\nabla\boldsymbol{t} = \kappa\boldsymbol{\nu}, \tag{99}$$

where τ is the torsion and κ the curvature. The unit normal $\boldsymbol{n} = \nabla p/|\nabla p|$ to the magnetic surface can be expressed as

$$\boldsymbol{n} = -\boldsymbol{b}\sin\phi + \boldsymbol{\nu}\cos\phi \tag{100}$$

since $\boldsymbol{t}\cdot\nabla p = 0$, where ϕ is the angle between the osculating plane of the line of force and the tangent plane to the magnetic surface. If one writes, since $\boldsymbol{n}\cdot\nabla\times\boldsymbol{B} = 0$,

$$\nabla\times\boldsymbol{B} = |\nabla\times\boldsymbol{B}|(\boldsymbol{t}\cos\theta + \boldsymbol{n}\times\boldsymbol{t}\sin\theta), \tag{101}$$

it is readily established that

$$-(\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{n}) = \kappa\boldsymbol{\nu}\cdot\nabla p + (\boldsymbol{B}\cdot\nabla\times\boldsymbol{B})(\tau - \boldsymbol{t}\cdot\nabla\phi). \tag{102}$$

It follows directly from (99) that if the line of force in question is bending in the direction of the pressure gradient then $\kappa\boldsymbol{\nu}\cdot\nabla p < 0$. Moreover $\tau - \boldsymbol{t}\cdot\nabla\phi$ measures the twist of $\boldsymbol{n}$ relative to $\boldsymbol{\nu}$, while the signature of $\boldsymbol{B}\cdot\nabla\times\boldsymbol{B}$ depends on the direction of the current density $\boldsymbol{J} = (c/4\pi)\nabla\times\boldsymbol{B}$ relative to the magnetic field.

Suppose that either by use of a trial function or by analytic or numerical minimization of (98) one finds a ξ such that $W < 0$ with $\boldsymbol{n}\cdot\xi = 0$ on S. Then, clearly, the system is unstable. If it can be shown that $W > 0$ with $\boldsymbol{n}\cdot\xi = 0$ on S it is necessary to relax this boundary condition and seek to minimize $W_V + W_S + W_F$. The enhanced potential for instability arises because W_S is possibly negative, though W_V is intrinsically non-negative.

3.1.6. The example of a magnetic field-free plasma supported by a vacuum magnetic field

A particularly simple application of the energy principle is to the case of a system made up of a plasma in which $\boldsymbol{B} = 0$, p = constant surrounded by a vacuum magnetic field. In this case

$$W_F = \frac{1}{2}\int d^3r\, p(\nabla\cdot\xi)^2 \geq 0.$$

Choose $\boldsymbol{\xi}$ so that $\nabla\cdot\boldsymbol{\xi} = 0$. For any given assignment of $\boldsymbol{n}\cdot\boldsymbol{\xi}$ and S one can find such a $\boldsymbol{\xi}$ by writing $\boldsymbol{\xi} = \nabla\phi$ and solving $\nabla^2\phi = 0$. Then

$$W = \frac{1}{2}\int_S d^2r(\hat{\boldsymbol{n}}\cdot\xi)^2\hat{\boldsymbol{n}}\cdot\nabla\frac{\hat{B}^2}{8\pi} + \int_{II} d^3r(\nabla\times A)^2, \tag{103}$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the interface S pointing towards the plasma. Let $\boldsymbol{R}$ be the vector pointing from the point on the line of force in question to the associated center of curvature. Now since p = constant in the plasma, it follows from (21) that $\hat{B}^2$ = constant on S. Moreover $\nabla\times\hat{\boldsymbol{B}} = 0$ whence with $\hat{\boldsymbol{B}} = \hat{B}\boldsymbol{t}$

$$\tfrac{1}{2}\hat{\boldsymbol{n}}\cdot\nabla\hat{B}^2 = \hat{\boldsymbol{n}}\cdot\left[\hat{\boldsymbol{B}}\times(\nabla\times\hat{\boldsymbol{B}}) + \hat{\boldsymbol{B}}\cdot\nabla\hat{\boldsymbol{B}}\right] = \hat{B}^2\hat{\boldsymbol{n}}\cdot(\boldsymbol{t}\cdot\nabla\boldsymbol{t}) = \hat{\boldsymbol{n}}\cdot\boldsymbol{R}\hat{B}^2/R^2 \tag{104}$$

since consequent to (99) $\boldsymbol{t}\cdot\nabla\boldsymbol{t} = \kappa\boldsymbol{\nu} = \boldsymbol{R}/R^2$, and $\hat{\boldsymbol{n}}\cdot\boldsymbol{t} = 0$.

Consider a point on S where $\boldsymbol{R}$ is directed toward the plasma and construct there a local Cartesian coordinate system with z-axis normal to the surface and pointing into the vacuum, and the x-axis parallel to $\boldsymbol{B}$. Choose

$$\xi_z(x,y,0) = \xi_0 f(x,y)\sin ky \tag{105}$$

where f is a function of order unity in magnitude which falls to zero in the small distance $a \ll R$, and $ka^2 \gg R$. Choose also the trial vector potential

$$A = f(x,y)\,\nabla\left(\frac{\xi_0\hat{B}}{k}\cos ky\, e^{-kz}\right). \tag{106}$$

Note that with $\hat{\boldsymbol{B}} = \hat{B}\boldsymbol{e}_x$, (105) and (106) satisfy boundary condition (58). These choices of $\boldsymbol{\xi}$ and $\boldsymbol{A}$ make the vacuum contribution to (103) negligible compared with the surface contribution, for

$$\begin{aligned}\int_{II} d^3r(\nabla\times A)^2 &= \int_{II} d^3r\left[(\nabla f)\times\nabla\left(\frac{\xi_0\hat{B}}{k}\cos ky\, e^{-kz}\right)\right]^2\\ &\approx \int_{II} d^3r|\nabla f|^2\xi_0^2\hat{B}^2 e^{-2kz}\\ &\approx \xi_0^2\hat{B}^2/2k,\end{aligned} \tag{107}$$

since $|\nabla f|^2 \approx 1/a^2$ and $\int_S d^2r|\nabla f|^2 \approx a^2(\nabla f)^2 \approx 1$, while

$$\int_S d^2r(\hat{\boldsymbol{n}}\cdot\xi)^2\frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{R}\hat{B}^2}{R^2} \approx \frac{\xi_0^2\hat{B}^2a^2}{R}. \tag{108}$$

Therefore W is negative and the system is unstable. Note that this conclusion of instability holds if anywhere on the boundary the surface S is concave towards the plasma.

3.1.7. The diffuse linear pinch

The next example to be considered is that of a diffuse linear pinch (Newcomb, 1960). This is a model where in cylindrical coordinates r,θ,z one has $p = p(r)$ and $\boldsymbol{B} = B_\theta(r)\boldsymbol{e}_\theta + B_z(r)\boldsymbol{e}_z$. Then the condition of magnetostatic equilibrium (29) requires that

$$4\pi\frac{dp}{dr} + B_z\frac{dB_z}{dr} + \frac{B_\theta}{r}\frac{d(rB_\theta)}{dr} = 0 \tag{109}$$

and in the vacuum annulus surrounding the cylindrical plasma column of radius a

$$\hat{\boldsymbol{B}} = \hat{B}_\theta(a)(R/r)\boldsymbol{e}_\theta + \hat{B}_z\boldsymbol{e}_z \tag{110}$$

where B_z = constant, and following (31)

$$8\pi p(a) + B_\theta(a)^2 + B_z(a)^2 = \hat{B}_\theta(a)^2 + \hat{B}_z^2. \tag{111}$$

Because of the cylindrical symmetry one can seek solutions of the form

$$\xi = \sum_m \int dk\left[\xi_r(r)\boldsymbol{e}_r + \xi_\theta(r)\boldsymbol{e}_\theta + \xi_z(r)\boldsymbol{e}_z\right]e^{i(kz+m\theta)}. \tag{112}$$

Then on introducing

$$\xi = \xi_r, \qquad \eta = (im/r)\xi_\theta + ik\xi_z, \qquad \zeta = i(\xi_\theta B_z - \xi_z B_\theta) \tag{113}$$

with inverse

$$\xi_\theta = -i\frac{kr\zeta + rB_\theta\eta}{krB_z + mB_\theta}, \qquad \xi_z = i\frac{m\zeta - rB_z\eta}{krB_z + mB_\theta}. \tag{114}$$

it can be shown after some algebra that

$$W_F = \frac{\pi}{2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, W_{k,m}. \tag{115}$$

When $k \neq 0$ and $m \neq 0$, on using (98) one can write

$$W_{k,m} = \int dr\, r\left(\Lambda + 4\pi\gamma p\left|\eta + \frac{1}{r}\frac{d(r\xi)}{dr}\right|^2 + \frac{k^2r^2 + m^2}{r^2}|\zeta - \nu|^2\right) \tag{116}$$

with

$$\begin{aligned}\Lambda(\xi) = &\frac{1}{k^2r^2 + m^2}\left|(krB_z + mB_\theta)\frac{d\xi}{dr} + (krB_z - mB_\theta)\frac{\xi}{r}\right|^2\\ &+ \left((krB_z + mB_\theta)^2 - 2B_\theta\frac{d(rB_\theta)}{dr}\right)\frac{|\xi|^2}{r^2}\end{aligned} \tag{117}$$

$$\nu = \frac{r}{k^2r^2 + m^2}\left((krB_\theta - mB_z)\frac{d\xi}{dr} - (krB_\theta + mB_z)\frac{\xi}{r}\right). \tag{118}$$

When $m = 0$, $k \neq 0$ it is convenient to define

$$W_{k,0} = W_0 + \frac{1}{2}k^2\int_0^a dr\, rB_z^2|\xi|^2 \tag{119}$$

where

$$W_0=\int_0^a \mathrm{d}r\left[rB_z^2\left|\frac{\mathrm{d}\xi}{\mathrm{d}r}\right|^2+\left(\frac{B_z^2}{r}+2\frac{\mathrm{d}p}{\mathrm{d}r}\right)|\xi|^2\right]. \tag{120}$$

Finally

$$W_{0,0}=W_0+\int_0^a \mathrm{d}r\,r\left(B_\theta^2\left|\frac{\mathrm{d}\xi}{\mathrm{d}r}-\frac{\xi}{r}\right|^2+\gamma p\left|\frac{\mathrm{d}\xi}{\mathrm{d}r}+\frac{\xi}{r}\right|^2\right). \tag{121}$$

Clearly if any of the $W_{k,m}$ can be made negative the system is unstable. But when k and m are both nonzero it follows from (116) that $W_{k,m}$ is minimized by choosing

$$\eta+\frac{1}{r}\frac{\mathrm{d}(r\boldsymbol{\xi})}{\mathrm{d}r}=0, \qquad \zeta-\nu=0.$$

Moreover, writing $\xi=\alpha+\mathrm{i}\beta$, then $W_{k,m}(\xi)=W_{k,m}(\alpha)+W_{k,m}(\beta)$. Since α and β are real and independent, it is sufficient to minimize $W_{k,m}(\xi)$ treating ξ as real. For this purpose it is convenient to integrate by parts and write

$$W_{k,m}=\int \mathrm{d}r\left[f\left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)^2+g\xi^2\right] \tag{122}$$

where

$$f=\frac{r(krB_z+mB_\theta)^2}{k^2r^2+m^2} \tag{123}$$

$$g=\frac{1}{r}\frac{(krB_z-mB_\theta)^2}{k^2r^2+m^2}+\frac{(krB_z+mB_\theta)^2}{r}-\frac{2B_\theta}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r}-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{k^2r^2B_z^2-m^2B_\theta^2}{k^2r^2+m^2}\right). \tag{124}$$

Consider the case $m\neq 0$ and write $k=mq$. Then only the second term on the right in (124) depends on m; indeed it is positive and proportional to m^2. Thus the least stable case corresponds to $m^2=1$ and since $W_{k,m}=W_{-k,-m}$ it is sufficient to consider only $m=1$ for $-\infty<k<\infty$. Moreover, when $m=0$, $k\neq 0$, (119) indicates that $W_{k,0}$ is non-negative if W_0 (which does not contain k) is non-negative, since the second term on the right is manifestly non-negative. Thus it is only necessary to consider the limit $k\to 0$. Finally, when $k=0$ it is clear from (121) that $W_{0,0}\geqslant W_0$. Thus the system is stable against perturbations with $m=0$ if $W_0>0$. Note that W_0 has the same form as (122).

A complete and rigorous analysis of the stability of the diffuse linear pinch has been given by Newcomb (1960). Here a simple sufficient condition for instability will be derived, and the balance of Newcomb's results will merely be quoted. To this end note that if $W_{k,m}$ of (122) can be made negative by a trial function $\xi_0(r)$, then it can be made negative by employing $A\xi_0(r)$ when A is an arbitrarily large factor. Thus in order to obtain a well posed minimum problem one must impose a norm to bound (122) from below. In particular trial functions will be treated which are localized near the zeros of the coefficient f which vanishes at points r_s such that

$$kr_sB_z(r_s)+mB_\theta(r_s)=0. \tag{125}$$

In the neighborhood of r_s, f and g can be well approximated by the leading non-identically zero terms in their Taylor series, namely

$$f=\alpha(r-r_s)^2, \qquad g=-\beta, \tag{126}$$

where the constants α and β are given by

$$\alpha=\left.\frac{r^2B_\theta^2B_z^2}{B^2}\left(\frac{\mathrm{d}\ln\mu}{\mathrm{d}r}\right)^2\right|_{r=r_s}\geqslant 0 \tag{127}$$

$$\beta=\left.-\frac{8\pi B_\theta^2}{B^2}\frac{\mathrm{d}p}{\mathrm{d}r}\right|_{r=r_s} \tag{128}$$

and

$$\mu=\frac{mB_\theta}{krB_z}. \tag{129}$$

Since in most cases of interest $\mathrm{d}p/\mathrm{d}r\leqslant 0$ one has $\beta>0$. For perturbations localized in a neighborhood sufficiently close to $r=r_s$ one can write for perturbations that vanish at r_1 and r_2

$$W_{k,m}=\int_{r_1}^{r_2}\mathrm{d}r\left[\alpha(r-r_s)^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)-\beta\xi^2\right]. \tag{130}$$

Suppose both α and β are positive and minimize

$$\int_{r_1}^{r_2}\mathrm{d}r(r-r_s)^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) \tag{131}$$

subject to the norm

$$\int_{r_1}^{r_2}\mathrm{d}r\,\xi^2=1. \tag{132}$$

By the standard techniques of the calculus of variations this leads to the Euler–Lagrange equation, homogeneous in $r-r_s$,

$$0=\frac{\mathrm{d}}{\mathrm{d}r}\left((r-r_s)^2\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)+(\lambda^2+\tfrac{1}{4})\xi, \qquad \xi(r_1)=0,\ \xi(r_2)=0, \tag{133}$$

where λ^2 is a Lagrange multiplier. Equation (133) has the solution

$$\xi=A(r-r_s)^{-1/2}\sin\left(\lambda\ln\frac{r_2-r_s}{r-r_s}\right), \tag{134}$$

where A is a normalizing constant, provided that

$$\lambda=n\pi\left[\ln\frac{r_2-r_s}{r_1-r_s}\right]^{-1}, \qquad n=1,2,3,4,\ldots \tag{135}$$

Note that one can make λ arbitrarily small by choosing r_1 sufficiently close to r_s. If (139) with $n=1$ is used as a trial function in (130), there results on integration by

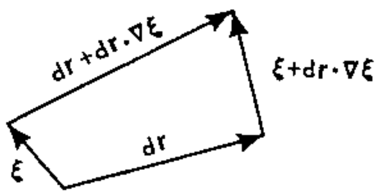

Fig. 3.1.7. Diagram relating the change in the vector element of length $d\boldsymbol{r}$ under an infinitesimal displacement $\boldsymbol{\xi}$.

parts

$$W_{k,m}=\int_{r_1}^{r_2}dx\left[\frac{d}{dr}\left(\alpha(r-r_s)^2\frac{d\xi}{dr}\xi\right)-\alpha\xi\frac{d}{dr}\left((r-r_s)^2\frac{d\xi}{dr}\right)-\beta\xi^2\right] \tag{136}$$

on employing (132) and (133), (136) can be reduced to

$$W_{k,m}=\alpha\left(\lambda^2+\tfrac{1}{4}\right)-\beta. \tag{137}$$

Since λ can be made arbitrarily small, $W_{k,m}$ can be made negative if

$$\beta>\tfrac{1}{4}\alpha. \tag{138}$$

Equation (138) is clearly a sufficient condition for instability, and with the inequality sign reversed is a necessary condition for stability. This is called Suydam's criterion (Suydam, 1958). Note that it is independent of k and m, and local to those points where (125) holds.

Newcomb has also shown that the diffuse linear pinch is unstable if there is any solution of

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right)-g\xi=0 \tag{139}$$

perhaps singular at points where (125) holds but such that $W_{k,m}$ is defined, which vanishes in any interval $r_s<r<r_{s+1}$. Moreover it is necessary for stability that the plasma pressure have a minimum at $r=0$.

Appendices

3.1.A. Various vector theorems

Consider a vector element of length $d\boldsymbol{r}$ that is displaced by an infinitesimal displacement field $\boldsymbol{\xi}$ which depends on $\boldsymbol{r}$. Then the change in $d\boldsymbol{r}$ under displacement is readily seen from Fig. 3.1.7 to be

$$\delta\, d\boldsymbol{r}=d\boldsymbol{r}\cdot\nabla\boldsymbol{\xi}. \tag{A1}$$

Let $d\boldsymbol{r}'$ be another element of length not collinear with $d\boldsymbol{r}$. Then one can define the vector element of area associated with the trapezoid which has $d\boldsymbol{r}$ and $d\boldsymbol{r}'$ as sides to

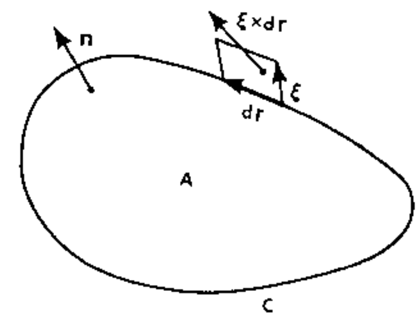

Fig. 3.1.8. Diagram of the change of an area A bounded by a curve C associated with the displacement $\boldsymbol{\xi}$ of an element of length $d\boldsymbol{r}$ in C.

be

$$d^2\boldsymbol{r}=d\boldsymbol{r}\times d\boldsymbol{r}'. \tag{A2}$$

The change in $d^2\boldsymbol{r}$ under displacement correct to linear order in $\boldsymbol{\xi}$ is readily calculated to be

$$\begin{aligned}\delta d^2\boldsymbol{r}&=(d\boldsymbol{r}\cdot\nabla\boldsymbol{\xi})\times d\boldsymbol{r}'+d\boldsymbol{r}\times(d\boldsymbol{r}'\cdot\nabla\boldsymbol{\xi})\\&=-\left[(d\boldsymbol{r}\times d\boldsymbol{r}')\times\nabla\right]\times\boldsymbol{\xi}.\end{aligned} \tag{A3}$$

If $\boldsymbol{n}$ is the unit vector parallel to $d^2\boldsymbol{r}=d^2r\,\boldsymbol{n}$, (A3) implies

$$d^2r\,\delta\boldsymbol{n}+\boldsymbol{n}\,\delta d^2r=-d^2r(\boldsymbol{n}\times\nabla)\times\boldsymbol{\xi}. \tag{A4}$$

Since $\boldsymbol{n}\cdot\boldsymbol{n}=1$, it follows that $\boldsymbol{n}\cdot\delta\boldsymbol{n}=0$, and the scalar product of (A4) with $\boldsymbol{n}$ yields

$$\delta d^2r=-d^2r\,\boldsymbol{n}\cdot(\boldsymbol{n}\times\nabla)\times\boldsymbol{\xi}=-d^2r\,\boldsymbol{n}\times(\boldsymbol{n}\times\nabla)\cdot\boldsymbol{\xi}. \tag{A5}$$

If one subtracts $\boldsymbol{n}$ times (A5) from (A4) there results

$$\begin{aligned}\delta\boldsymbol{n}&=-(\boldsymbol{n}\times\nabla)\times\boldsymbol{\xi}+\boldsymbol{n}\boldsymbol{n}\times(\boldsymbol{n}\times\nabla)\cdot\boldsymbol{\xi}\\&=-(\nabla\boldsymbol{\xi})\cdot\boldsymbol{n}+\boldsymbol{n}\boldsymbol{n}\cdot(\nabla\boldsymbol{\xi})\cdot\boldsymbol{n}.\end{aligned} \tag{A6}$$

Note that (A5) and (A6) involve only $\boldsymbol{n}\times\nabla$, that is derivatives tangent to the surface. Moreover if $\boldsymbol{\xi}$ is derived from a flow field described by velocity $\boldsymbol{v}$, then $\boldsymbol{\xi}=\boldsymbol{v}\delta t$ and (A6) yields

$$\dot{\boldsymbol{n}}=\lim_{\delta t\to 0}\frac{\delta\boldsymbol{n}}{\delta t}=-(\nabla\boldsymbol{v})\cdot\boldsymbol{n}+\boldsymbol{n}\boldsymbol{n}\cdot(\nabla\boldsymbol{v})\cdot\boldsymbol{n} \tag{A7}$$

while (A3) implies

$$(d^2\boldsymbol{r})^{\cdot}=-(d^2\boldsymbol{r}\times\nabla)\times\boldsymbol{v}. \tag{A8}$$

Consider the change in area associated with the displacement of the bounding curve C of an area A. The contribution of a segment $d\boldsymbol{r}$ along C to this change is seen from Fig. 3.1.8 to be

$$\delta\,\mathrm{d}^2\boldsymbol{r} = \boldsymbol{\xi}\times\mathrm{d}\boldsymbol{r}. \tag{A9}$$

Thus the change in the magnetic flux through A is

$$\delta\int_A \mathrm{d}^2\boldsymbol{r}\cdot\boldsymbol{B} = \int_A \mathrm{d}^2\boldsymbol{r}\cdot\delta\boldsymbol{B} + \int_C \boldsymbol{B}\cdot\boldsymbol{\xi}\times\mathrm{d}\boldsymbol{r} \tag{A10}$$

which implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathrm{d}^2\boldsymbol{r}\cdot\boldsymbol{B} &= \int_A \mathrm{d}^2\boldsymbol{r}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \int_C \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{v}\times\boldsymbol{B}\\ &= \int_A \mathrm{d}^2\boldsymbol{r}\cdot\left(\frac{\partial\boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B})\right).\end{aligned} \tag{A11}$$

It is wished to demonstrate that

$$\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) = (\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{n}\cdot\boldsymbol{\xi}\boldsymbol{n}\times\boldsymbol{B}), \tag{A12}$$

that is, it involves only tangential derivatives of the normal component of $\boldsymbol{\xi}$. To this end write, assuming $\boldsymbol{n}\cdot\boldsymbol{B}=0$,

$$\begin{aligned}\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) &= \boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\nabla\cdot\boldsymbol{\xi})\\ &= \boldsymbol{B}\cdot(\nabla\boldsymbol{\xi})\cdot\boldsymbol{n} - \boldsymbol{\xi}\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{n}\\ &= \boldsymbol{B}\cdot\nabla(\boldsymbol{\xi}\cdot\boldsymbol{n}) - \boldsymbol{B}\cdot(\nabla\boldsymbol{n})\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla(\boldsymbol{n}\cdot\boldsymbol{B}) + \boldsymbol{\xi}\cdot(\nabla\boldsymbol{n})\cdot\boldsymbol{B}.\end{aligned} \tag{A13}$$

But since $\boldsymbol{n}$ is the unit normal to S, it can be assumed that S is a level surface of a function f, whence if $c = |\nabla f|^{-1}$

$$\boldsymbol{n} = c\nabla f \tag{A14}$$

and

$$\begin{aligned}\nabla\boldsymbol{n} &= c\nabla\nabla f + (\nabla c)\nabla f\\ &= c\nabla\nabla f + c^{-1}(\nabla c)\boldsymbol{n}\end{aligned} \tag{A15}$$

whence as $\nabla\nabla f$ is symmetric, since $\boldsymbol{n}\cdot\boldsymbol{B}=0$,

$$\boldsymbol{\xi}\cdot(\nabla\boldsymbol{n})\cdot\boldsymbol{B} - \boldsymbol{B}\cdot(\nabla\boldsymbol{n})\cdot\boldsymbol{\xi} = -c\boldsymbol{B}\cdot(\nabla c)\boldsymbol{n}\cdot\boldsymbol{\xi} \tag{A16}$$

and $\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})$ involves only $\boldsymbol{n}\cdot\boldsymbol{\xi}$. Hence, writing $\boldsymbol{\xi} = \boldsymbol{n}\boldsymbol{n}\cdot\boldsymbol{\xi} + \boldsymbol{n}\times(\boldsymbol{\xi}\times\boldsymbol{n})$ and inserting it in $\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})$, only the part involving $\boldsymbol{n}\cdot\boldsymbol{\xi}$ can survive, whence

$$\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) = (\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{n}\cdot\boldsymbol{\xi}\boldsymbol{n}\times\boldsymbol{B}). \tag{A17}$$

Note that (A15) implies $\nabla\times\boldsymbol{n} = c\nabla\times(\nabla f) + c^{-1}(\nabla c)\times\boldsymbol{n} = c^{-1}(\nabla c)\times\boldsymbol{n}$ whence

$$\boldsymbol{n}\cdot\nabla\times\boldsymbol{n} = 0. \tag{A18}$$

3.1.B. A vector identity

Recall that $\nabla p = (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ and that $\boldsymbol{Q} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})$. Then the vector

$$\begin{aligned}&\nabla(\boldsymbol{\xi}\cdot\nabla p) + (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}\\ &\quad= (\nabla\boldsymbol{\xi})\cdot\nabla p + \boldsymbol{\xi}\cdot\nabla\nabla p + (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}\\ &\quad= [(\nabla p)\times\nabla]\times\boldsymbol{\xi} + (\nabla\cdot\boldsymbol{\xi})\nabla p + \boldsymbol{\xi}\cdot\nabla\nabla p + (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}\\ &\quad= (1/4\pi)\{[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]\times\nabla)\times\boldsymbol{\xi} + (\nabla\cdot\boldsymbol{\xi})\nabla p\\ &\qquad + \boldsymbol{\xi}\cdot\nabla\nabla p + (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}\\ &\quad= (1/4\pi)[\boldsymbol{B}(\nabla\times\boldsymbol{B})\cdot\nabla - (\nabla\times\boldsymbol{B})\boldsymbol{B}\cdot\nabla]\times\boldsymbol{\xi} + \nabla\cdot\boldsymbol{\xi}\nabla p\\ &\qquad + \boldsymbol{\xi}\cdot\nabla\nabla p + (1/4\pi)(\nabla\times\boldsymbol{B})\times[\boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\nabla\cdot\boldsymbol{\xi}]\\ &\quad= (1/4\pi)\{\boldsymbol{B}\times[(\nabla\times\boldsymbol{B})\cdot\nabla\boldsymbol{\xi}] - \boldsymbol{B}\times[\boldsymbol{\xi}\cdot\nabla(\nabla\times\boldsymbol{B})]\\ &\qquad - \boldsymbol{\xi}\cdot\nabla[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}] + \boldsymbol{B}\times(\nabla\times\boldsymbol{B})\nabla\cdot\boldsymbol{\xi}\}\\ &\qquad + (\nabla\cdot\boldsymbol{\xi})\nabla p + \boldsymbol{\xi}\cdot\nabla\nabla p\\ &\quad= (1/4\pi)\boldsymbol{B}\times\{\nabla\times[\boldsymbol{\xi}\times(\nabla\times\boldsymbol{B})]\} + (1/4\pi)\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\nabla\cdot\boldsymbol{\xi}\\ &\quad= (1/4\pi)\boldsymbol{B}\times\{\nabla\times[\boldsymbol{\xi}\times(\nabla\times\boldsymbol{B})]\} - (\nabla\cdot\boldsymbol{\xi})\nabla p.\end{aligned} \tag{B1}$$

3.1.C. Details of the derivation of conservation of energy and self adjointness

Let $\boldsymbol{Q}' = \nabla\times(\boldsymbol{\xi}'\times\boldsymbol{B})$ and $\boldsymbol{Q} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})$. Then on using (B1)

$$\begin{aligned}\boldsymbol{\xi}'\cdot\boldsymbol{F}(\boldsymbol{\xi}) &= \boldsymbol{\xi}'\cdot[\nabla(\gamma p\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p) + (1/4\pi)(\nabla\times\boldsymbol{Q})\times\boldsymbol{B}\\ &\qquad + (1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}]\\ &= \boldsymbol{\xi}'\cdot[\nabla(\gamma p\nabla\cdot\boldsymbol{\xi}) + (1/4\pi)(\nabla\times\boldsymbol{Q})\times\boldsymbol{B}\\ &\qquad + (1/4\pi)\boldsymbol{B}\times\{\nabla\times[\boldsymbol{\xi}\times(\nabla\times\boldsymbol{B})]\} - (\nabla\cdot\boldsymbol{\xi})\nabla p]\\ &= \nabla\cdot\{\boldsymbol{\xi}'\gamma p\nabla\cdot\boldsymbol{\xi} - (1/4\pi)\boldsymbol{Q}\times(\boldsymbol{\xi}'\times\boldsymbol{B})\\ &\qquad + (1/4\pi)[\boldsymbol{\xi}\times(\nabla\times\boldsymbol{B})]\times(\boldsymbol{\xi}'\times\boldsymbol{B})]\}\\ &\quad - \gamma p(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\xi}') - (1/4\pi)\boldsymbol{Q}\cdot\boldsymbol{Q}'\\ &\quad + (1/4\pi)[\boldsymbol{\xi}\times(\nabla\times\boldsymbol{B})]\cdot\boldsymbol{Q}' - (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi}'\cdot\nabla p\\ &= \nabla\cdot\{\boldsymbol{\xi}'\gamma p\nabla\cdot\boldsymbol{\xi} - (1/4\pi)\boldsymbol{Q}\times(\boldsymbol{\xi}'\times\boldsymbol{B})\\ &\qquad + \boldsymbol{\xi}'(1/4\pi)\boldsymbol{\xi}\cdot(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - (1/4\pi)\boldsymbol{B}\boldsymbol{\xi}'\times\boldsymbol{\xi}\cdot\nabla\times\boldsymbol{B}\}\\ &\quad - (1/4\pi)\boldsymbol{Q}\cdot\boldsymbol{Q}' + \boldsymbol{\xi}\cdot(1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}' - (\nabla\cdot\boldsymbol{\xi})(\gamma p\nabla\cdot\boldsymbol{\xi}' + \boldsymbol{\xi}'\cdot\nabla p)\\ &= \nabla\cdot\{\boldsymbol{\xi}'\gamma p\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\gamma p\nabla\cdot\boldsymbol{\xi}' - (1/4\pi)\boldsymbol{Q}\times(\boldsymbol{\xi}'\times\boldsymbol{B})\\ &\qquad + (1/4\pi)\boldsymbol{Q}'\times(\boldsymbol{\xi}\times\boldsymbol{B}) + \boldsymbol{\xi}'\boldsymbol{\xi}\cdot\nabla p - \boldsymbol{\xi}\boldsymbol{\xi}'\cdot\nabla p - \boldsymbol{B}\boldsymbol{\xi}'\times\boldsymbol{\xi}\cdot\nabla p\}\\ &\quad + \nabla\cdot[\boldsymbol{\xi}(\gamma p\nabla\cdot\boldsymbol{\xi}' + \boldsymbol{\xi}'\cdot\nabla p) - (1/4\pi)\boldsymbol{Q}'\times(\boldsymbol{\xi}\times\boldsymbol{B})]\end{aligned}$$

$$-(1/4\pi)\boldsymbol{Q}\cdot\boldsymbol{Q}'+\boldsymbol{\xi}\cdot(1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}'$$
$$-(\nabla\cdot\boldsymbol{\xi})(\gamma p\,\nabla\cdot\boldsymbol{\xi}'+\boldsymbol{\xi}'\cdot\nabla p) \tag{C1}$$

or

$$\begin{aligned}\boldsymbol{\xi}'\cdot\boldsymbol{F}(\boldsymbol{\xi}) = {} & \nabla\cdot\{\boldsymbol{\xi}'[\gamma p\,\nabla\cdot\boldsymbol{\xi}-(1/4\pi)\boldsymbol{Q}\cdot\boldsymbol{B}+\boldsymbol{\xi}\cdot\nabla p]\\ & +\boldsymbol{B}((1/4\pi)(\boldsymbol{\xi}'\cdot\boldsymbol{Q}-\boldsymbol{\xi}\cdot\boldsymbol{Q}')+\boldsymbol{\xi}\times\boldsymbol{\xi}'\cdot\nabla p)\\ & -\boldsymbol{\xi}[\gamma p\,\nabla\cdot\boldsymbol{\xi}'-(1/4\pi)\boldsymbol{Q}'\cdot\boldsymbol{B}+\boldsymbol{\xi}'\cdot\nabla p]\}\\ & +\boldsymbol{\xi}\cdot[\nabla(\gamma p\,\nabla\cdot\boldsymbol{\xi}'+\boldsymbol{\xi}'\cdot\nabla p)+(1/4\pi)(\nabla\times\boldsymbol{B})\times\boldsymbol{Q}'\\ & +(1/4\pi)(\nabla\times\boldsymbol{Q}')\times\boldsymbol{B}]\\ = {} & \nabla\cdot\{\boldsymbol{\xi}'[\gamma p\,\nabla\cdot\boldsymbol{\xi}-(1/4\pi)(\boldsymbol{Q}\cdot\boldsymbol{B}+\boldsymbol{\xi}\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{B})+\boldsymbol{\xi}\cdot\nabla(p+B^2/8\pi)]\\ & -\boldsymbol{\xi}[\gamma p\,\nabla\cdot\boldsymbol{\xi}'-(1/4\pi)(\boldsymbol{Q}'\cdot\boldsymbol{B}+\boldsymbol{\xi}'\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{B})+\boldsymbol{\xi}'\cdot\nabla(p+B^2/8\pi)]\\ & +\boldsymbol{B}((1/4\pi)(\boldsymbol{\xi}'\cdot\boldsymbol{Q}-\boldsymbol{\xi}\cdot\boldsymbol{Q}')+\boldsymbol{\xi}\times\boldsymbol{\xi}'\cdot\nabla p)\}+\boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi}').\end{aligned} \tag{C2}$$

Thus if (C2) is integrated over the volume of the plasma, since $\boldsymbol{n}\cdot\boldsymbol{B}=0$ on S

$$\begin{aligned}&\int \mathrm{d}^3r[\boldsymbol{\xi}'\cdot\boldsymbol{F}(\boldsymbol{\xi})-\boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi}')]\\ &=\int \mathrm{d}^2r\,\boldsymbol{n}\cdot\left\{\boldsymbol{\xi}'\left[\gamma p\,\nabla\cdot\boldsymbol{\xi}-\frac{1}{4\pi}(\boldsymbol{Q}\cdot\boldsymbol{B}+\boldsymbol{\xi}\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{B})\right.\right.\\ &\qquad\left.+\boldsymbol{\xi}\cdot\nabla\left(p+\frac{B^2}{8\pi}-\frac{\hat{B}^2}{8\pi}\right)+\boldsymbol{\xi}\cdot\nabla\frac{\hat{\boldsymbol{B}}^2}{8\pi}\right]\\ &\qquad-\boldsymbol{\xi}\left[\gamma p\,\nabla\cdot\boldsymbol{\xi}'-\frac{1}{4\pi}(\boldsymbol{Q}'\cdot\boldsymbol{B}+\boldsymbol{\xi}'\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{B})\right.\\ &\qquad\left.\left.+\boldsymbol{\xi}'\cdot\nabla\left(p+\frac{B^2}{8\pi}-\frac{\hat{B}^2}{8\pi}\right)+\boldsymbol{\xi}'\cdot\nabla\frac{\hat{\boldsymbol{B}}^2}{8\pi}\right]\right\}.\end{aligned} \tag{C3}$$

Now it follows from (33) which holds everywhere on S that

$$\boldsymbol{n}\times\nabla\left(p+\frac{B^2}{8\pi}-\frac{\hat{B}^2}{8\pi}\right)=0. \tag{C4}$$

Thus the terms in (C3) involving $p+B^2/8\pi-\hat{B}^2/8\pi$ cancel. Moreover on using (58), (66), (49), and (54) (since $\hat{\boldsymbol{n}}=-\boldsymbol{n}$) (C3) can be rewritten as:

$$\begin{aligned}4\pi\int \mathrm{d}^3r[\boldsymbol{\xi}'\cdot\boldsymbol{F}(\boldsymbol{\xi})-\boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi}')] &= \int_S \mathrm{d}^2r\,\boldsymbol{n}\cdot\langle-\boldsymbol{\xi}'\hat{\boldsymbol{B}}\cdot\nabla\times\boldsymbol{A}+\boldsymbol{\xi}\hat{\boldsymbol{B}}\cdot\nabla\times\boldsymbol{A}'\rangle\\ &=-\int_{S+S'}\mathrm{d}^2r[\hat{\boldsymbol{n}}\times\boldsymbol{A}'\cdot\nabla\times\boldsymbol{A}-\hat{\boldsymbol{n}}\times\boldsymbol{A}\cdot\nabla\times\boldsymbol{A}']\\ &=-\int_{S+S'}\mathrm{d}^2\boldsymbol{r}\cdot[\boldsymbol{A}'\times(\nabla\times\boldsymbol{A})-\boldsymbol{A}\times(\nabla\times\boldsymbol{A}')]\\ &=-\int_{\mathrm{II}}\mathrm{d}^3r\,\nabla\cdot[\boldsymbol{A}'\times(\nabla\times\boldsymbol{A})-\boldsymbol{A}\times(\nabla\times\boldsymbol{A}')]\\ &=\int \mathrm{d}^3r[\boldsymbol{A}'\cdot\nabla\times\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\nabla\times\boldsymbol{A}']\\ &=0.\end{aligned} \tag{C5}$$

In (C5) $\boldsymbol{A}$ is the vector potential associated with $\boldsymbol{\xi}$ and $\boldsymbol{A}'$ that associated with $\boldsymbol{\xi}'$. Note that (C5) applies equally well where $\boldsymbol{\xi}'=\partial\boldsymbol{v}/\partial t$ since the equations and boundary conditions are the same, only with $\boldsymbol{v}$ and $\boldsymbol{A}$ replaced by $\partial\boldsymbol{v}/\partial t$ and $\partial\boldsymbol{A}/\partial t$.

3.1.D. Transformation of W_F

Note that in the fluid where $4\pi\nabla p=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ the vectors ∇p, $\boldsymbol{B}$, and $\nabla\times\boldsymbol{B}$ are noncoplanar and one can write

$$\boldsymbol{\xi}=a\,\nabla p+b\,\nabla\times\boldsymbol{B}+c\boldsymbol{B}, \tag{D1}$$

whence

$$\boldsymbol{\xi}\times\boldsymbol{B}=-a\boldsymbol{B}\times\nabla p+4\pi b\,\nabla p \tag{D2}$$

$$\boldsymbol{\xi}\times(\nabla\times\boldsymbol{B})=-a(\nabla\times\boldsymbol{B})\times\nabla p-4\pi c\,\nabla p. \tag{D3}$$

Moreover

$$\begin{aligned}&a(\nabla p)^2\,\nabla\cdot(b\,\nabla\times\boldsymbol{B}+c\boldsymbol{B})-\nabla\cdot\left[a(\nabla p)^2(b\,\nabla\times\boldsymbol{B}+c\boldsymbol{B})\right]\\ &=-(b\,\nabla\times\boldsymbol{B}+c\boldsymbol{B})\cdot\nabla\left[a(\nabla p)^2\right]\\ &=-b\,\nabla\cdot\left[a(\nabla p)^2\,\nabla\times\boldsymbol{B}\right]-c\,\nabla\cdot\left[a(\nabla p)^2\boldsymbol{B}\right]\\ &=-b\,\nabla\cdot\langle(\nabla p)\times[a(\nabla\times\boldsymbol{B})\times\nabla p]\rangle-c\,\nabla\cdot[(\nabla p)\times(a\boldsymbol{B}\times\nabla p)]\\ &=b(\nabla p)\cdot\nabla\times[a(\nabla\times\boldsymbol{B})\times\nabla p]+c(\nabla p)\cdot\nabla\times(a\boldsymbol{B}\times\nabla p)\\ &=\nabla\cdot\langle[a(\nabla\times\boldsymbol{B})\times\nabla p]\times b\,\nabla p\rangle+a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(b\,\nabla p)\\ &\quad+c(\nabla p)\cdot\nabla\times(a\boldsymbol{B}\times\nabla p),\end{aligned} \tag{D4}$$

whence

$$\begin{aligned}&a(\nabla p)^2\,\nabla\cdot(b\,\nabla\times\boldsymbol{B}+c\boldsymbol{B})\\ &=\nabla\cdot\left[ac(\nabla p)^2\boldsymbol{B}\right]+a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(b\,\nabla p)\\ &\quad+c(\nabla p)\cdot\nabla\times(a\boldsymbol{B}\times\nabla p).\end{aligned} \tag{D5}$$

Thus on using (86), Gauss's theorem, and $\boldsymbol{n}\cdot\boldsymbol{B}=0$,

$$\begin{aligned}8\pi W_F=\int_{\mathrm{I}}\mathrm{d}^3r\langle &4\pi\gamma p(\nabla\cdot\boldsymbol{\xi})^2+[\nabla\times(-a\boldsymbol{B}\times\nabla p+4\pi b\,\nabla p)]^2\\ &+4\pi a(\nabla p)^2\,\nabla\cdot(a\,\nabla p+b\,\nabla\times\boldsymbol{B}+c\boldsymbol{B})\\ &+[a(\nabla\times\boldsymbol{B})\times\nabla p+4\pi c\,\nabla p]\cdot\nabla\times(-a\boldsymbol{B}\times\nabla p+4\pi b\,\nabla p)\rangle\end{aligned}$$

$$= \int_{\mathrm{I}} \mathrm{d}^3 r \{4\pi\gamma p(\nabla\cdot\boldsymbol{\xi})^2 + [\nabla\times(-a\boldsymbol{B}\times\nabla p + 4\pi b\nabla p)]^2$$
$$+ 4\pi a(\nabla p)^2\nabla\cdot(a\nabla p)$$
$$+ 8\pi a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(b\nabla p)$$
$$- a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(a\boldsymbol{B}\times\nabla p)\}$$
$$= \int_{\mathrm{I}} \mathrm{d}^3 r \{4\pi\gamma p(\nabla\cdot\boldsymbol{\xi})^2 + [\nabla\times(-a\boldsymbol{B}\times\nabla p + 4\pi b\nabla p) + a(\nabla\times\boldsymbol{B})\times\nabla p]^2$$
$$+ a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(a\boldsymbol{B}\times\nabla p)$$
$$- a^2[(\nabla\times\boldsymbol{B})\times\nabla p]^2 + 4\pi a(\nabla p)^2\nabla\cdot(a\nabla p)\}. \tag{D6}$$

Define

$$\boldsymbol{n} = \nabla p/|\nabla p|. \tag{D7}$$

Then since $4\pi\nabla p = (\nabla\times\boldsymbol{B})\times\boldsymbol{B}$

$$-a^2[(\nabla\times\boldsymbol{B})\times\nabla p]^2 + a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(a\boldsymbol{B}\times\nabla p)$$
$$+ 4\pi a(\nabla p)^2\nabla\cdot(a\nabla p)$$
$$= -a^2(\nabla\times\boldsymbol{B})^2(\nabla p)^2 + a^2(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot\nabla\times(\boldsymbol{B}\times\nabla p)$$
$$+ a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot(\nabla a)\times(\boldsymbol{B}\times\nabla p)$$
$$+ 4\pi a^2(\nabla p)^2\nabla^2 p + 4\pi a(\nabla p)^2(\nabla a)\cdot\nabla p$$
$$= -a^2(\nabla\times\boldsymbol{B})^2(\nabla p)^2$$
$$+ a^2(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot[(\nabla p)\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\nabla p + \boldsymbol{B}\nabla^2 p]$$
$$+ a(\nabla\times\boldsymbol{B})\times(\nabla p)\cdot[\boldsymbol{B}(\nabla a)\cdot\nabla p - (\nabla p)\boldsymbol{B}\cdot\nabla a]$$
$$+ 4\pi a^2(\nabla p)^2\nabla^2 p + 4\pi a(\nabla p)^2(\nabla a)\cdot\nabla p$$
$$= a^2(\nabla p)^2\left[-(\nabla\times\boldsymbol{B})^2 + (\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{n})\right]. \tag{D8}$$

Since $\nabla\cdot\boldsymbol{B} = 0$, it follows that

$$0 = \nabla(\boldsymbol{n}\cdot\boldsymbol{B}) = \boldsymbol{n}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\boldsymbol{n} + \boldsymbol{n}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{n}), \tag{D9}$$

whence

$$-(\nabla\times\boldsymbol{B})^2 + (\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot[\boldsymbol{n}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{n}]$$
$$= -(\nabla\times\boldsymbol{B})^2 + (\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot[-2\boldsymbol{B}\cdot\nabla\boldsymbol{n} - \boldsymbol{n}\times(\nabla\times\boldsymbol{B}) - \boldsymbol{B}\times(\nabla\times\boldsymbol{n})]$$
$$= -(\nabla\times\boldsymbol{B})^2 + [\boldsymbol{n}\times(\nabla\times\boldsymbol{B})]^2 - 2(\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{n})$$
$$-(\nabla\times\boldsymbol{B})\cdot\boldsymbol{B}\boldsymbol{n}\cdot\nabla\times\boldsymbol{n} - (\nabla\times\boldsymbol{B})\cdot(\nabla\times\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{B}$$
$$= -2(\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{n}) \tag{D10}$$

since by virtue of $4\pi\nabla p = (\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ it follows that $\boldsymbol{n}\cdot\nabla\times\boldsymbol{B} = 0$, whence

$$[\boldsymbol{n}\times(\nabla\times\boldsymbol{B})]^2 = \boldsymbol{n}^2(\nabla\times\boldsymbol{B})^2 - (\boldsymbol{n}\cdot\nabla\times\boldsymbol{B})^2 = (\nabla\times\boldsymbol{B})^2, \tag{D11}$$

and since $\boldsymbol{n}$ is a unit vector, the integral over an arbitrary area in a surface p = constant of $\boldsymbol{n}\cdot\nabla\times\boldsymbol{n}$ yields on using Stokes' theorem

$$\int \mathrm{d}^2 r\,\boldsymbol{n}\cdot\nabla\times\boldsymbol{n} = \int \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{n} = 0 \tag{D12}$$

whence, since the surface is arbitrary, it follows that $\boldsymbol{n}\cdot\nabla\times\boldsymbol{n} = 0$. Thus one can write using (D6) and (D10)

$$8\pi W_{\mathrm{F}} = \int_{\mathrm{I}} \mathrm{d}^3 r\{4\pi\gamma p(\nabla\cdot\boldsymbol{\xi})^2 + [\boldsymbol{Q} + (\nabla\times\boldsymbol{B})\times\boldsymbol{n}\boldsymbol{n}\cdot\boldsymbol{\xi}]^2$$
$$-2(\boldsymbol{n}\cdot\boldsymbol{\xi})^2(\nabla\times\boldsymbol{B})\times\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{n})\}. \tag{D13}$$

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