→ Schematic Notes on Quasi-Modes

→ Should be Read as Supplement to [redacted]
Roberts-Taylor Paper

iv) Quasimodes and Relation to Resistive Interchanges

Ref.: K.V. Roberts and J.B. Taylor, Phys. Fluids 8 315 (1965).

→ Note: Salient features of resistive interchange

- localized to $\Delta/a \sim S^{-1/3}$ of $\underline{k}\cdot\underline{B}_0 = 0$ surface
- mode tied to resistive layer $\Delta/a \sim S^{-1/3}$

Contrast: Ideal interchange (shear-free system)

- global mode $\Delta/a \sim k_x a \sim 1$
- no special preferred location

→ Motivation:

- instabilities of interest due to transport caused (i.e. effect on confinement)
- mode space and time scales determine transport

$$D \sim (\Delta x)^2/\tau_c \sim \gamma/k_x^2$$

- for resistive g-mode, $k_x \sim \Delta^{-1} \sim S^{-1/3}$

$$D \sim \gamma / k_x^2 \sim \eta^{1/3}$$

$\downarrow$

global transport on resistive diffusion timescale (slow)

- thus, natural to seek resistive modes / wave-packets with $k_x a \sim 1$ (larger transport)

- intuition is that may be able to construct quasi-mode wave-packets with $\gamma \sim \eta^{1/3}$, $k_x a \sim 1$ i.e. radial structure of resistive-interchange determined by fourier analysis, as well as physics.

→ Approach:

- Consider sequence of small problems:

① ideal interchange + resistively coated end plates (localized resistivity ⟺ no line tying)

⇒ interchanging tubes can rotate, to adjust locally to shear

② distributed η, line-tying but no shear

⇒ η allows growing "slicing modes", with

$$\gamma \sim \eta k_y^2 \frac{g/L_p}{k_\parallel^2 V_A^2}$$

③ distributed η, no line-tying but finite shear

$\Rightarrow$ finite length (along B_0) wave-packet, with $k_x a \sim 1$. Here,

$$\gamma \sim \eta k_y^2 \frac{\gamma_I^2 L^2}{V_A^2}$$

with $\underline{L \sim \eta^{-1/3}}$ $\Rightarrow$ recovers $\gamma \sim \eta^{1/3}$

$\Rightarrow$ wave packet is superposition of localized g-modes with approximately identical growth rates.

i.e.

$\Rightarrow$ local orientation twists to align with shear

① Ideal Interchange with resistively coated plates

Recall, for ideal interchange

$$\gamma^2 \nabla_\perp^2 \vec{\phi} = V_A^2 \nabla_\parallel \nabla_\perp^2 \nabla_\parallel \vec{\phi} + \frac{g}{|L_p|} \frac{\partial^2}{\partial y^2} \vec{\phi}$$

$$\nabla_\parallel = \frac{\underline{k} \cdot \underline{B}_0}{|B_0|} \rightarrow k_y \frac{x}{L_s}$$

$$\nabla_\parallel = \left(\frac{\partial}{\partial z} + \frac{x}{L_s} \frac{\partial}{\partial y} \right)$$

Better to use "twisting coordinates": (aligned with shear)

$$\xi = x \qquad d\xi = dx$$

$$\chi = y - \frac{x}{L_s} z \qquad d\chi = dy - \frac{1}{L_s} (x\, dz + z\, dx)$$

$$\varphi = z \qquad d\varphi = dz$$

$$\frac{\partial}{\partial z} = \frac{d\chi}{dz} \frac{\partial}{\partial \chi} + \frac{d\xi}{dz} \frac{\partial}{\partial \xi} + \frac{d\varphi}{dz} \frac{\partial}{\partial \varphi}$$

$$= -\frac{x}{L_s} \frac{\partial}{\partial \chi} + 0 + \frac{\partial}{\partial \varphi}$$

$$\partial/\partial y = \partial/\partial x$$

$\Rightarrow$

$$\frac{\partial}{\partial z} + \frac{x}{L_s}\frac{\partial}{\partial y} = \frac{\partial}{\partial \varphi}$$

$\Rightarrow$

$$\gamma^2 \nabla_\perp^2 \tilde{\phi} = V_A^2 \frac{\partial}{\partial \varphi} \nabla_\perp^2 \frac{\partial}{\partial \varphi} \tilde{\phi} + \frac{g}{|L_p|}\frac{\partial^2}{\partial x^2}\tilde{\phi}$$

$$\phi = \phi^{(0)} + \phi^{(1)} + \cdots \quad , \qquad k_\parallel^2 V_A^2 \gg \gamma_I^2 \gtrsim \gamma^2$$

$\Rightarrow$

$$V_A^2 \frac{\partial}{\partial \varphi} \nabla_\perp^2 \frac{\partial}{\partial \varphi} \phi^{(0)} = 0 \qquad \Leftrightarrow \qquad D_0 \phi^{(0)} = 0$$

$$\gamma^2 \nabla_\perp^2 \tilde{\phi}^{(0)} = V_A^2 \frac{\partial}{\partial \varphi} \nabla_\perp^2 \frac{\partial}{\partial \varphi} \tilde{\phi}^{(1)} + \frac{g}{|L_p|}\frac{\partial^2}{\partial x^2}\phi^{(0)}$$

$\Leftrightarrow$

$$D_0 \phi^{(1)} = D_1 \phi^{(0)}$$

solvable <u>only</u> <u>if</u> $\displaystyle\int_{-L}^{L} \phi^{(0)} D_1 \phi^{(0)} d\varphi = 0$

-lgebra $\Rightarrow$

$$\frac{d^2\phi^{(0)}}{d\xi^2} + (A\xi^2 + B)\phi^{(0)} = 0$$

$$A = \frac{k_y^2}{L_s^2}\left(\frac{g}{L_p\gamma^2} - 1\right)$$

$$B = k_y^2\left(\frac{g}{L_p\gamma^2} - 1 - \frac{L^2}{3L_s^2}\right)$$

$\phi = 0$ at $x = \pm H$ $\Rightarrow$ $\frac{1}{\phi^{(0)}}\frac{\partial^2\phi^{(0)}}{\partial\xi^2} < 0$

$$\Rightarrow \left(\frac{\gamma_I^2}{\gamma^2} - 1\right) > \frac{1}{3}\frac{L^2}{L_s^2} \Big/ 1 + \frac{H^2}{L_s^2}$$

i.e

- L_s slows interchange growth, but does not stabilize interchange

$\Downarrow$

- consequence of resistivity localized to end plates

(5) Shear - Free + Line-Tying

Recall:

- $\phi \sim \cos\frac{m\pi z}{2L}\ \sin\frac{\pi x}{2H}\ e^{ik_y y}$ (b.c.'s)

$\Rightarrow$

$$\gamma^2 + \frac{k_z^2 V_A^2 \gamma}{\gamma + \eta k_\perp^2} - \frac{g k_y^2}{|L_\rho| k_\perp^2} = 0 \qquad \text{(local dispn.)}$$

$$k_\perp^2 = k_x^2 + k_y^2$$

Now, consistent with motivation:

- $k_x \ll k_y \Rightarrow k_\perp \cong k_y$
- $k_z \cong \pi/L$

Further:

- $\eta k_\perp^2 > \gamma$
- $k_z^2 V_A^2 \gg \gamma\ \eta k_y^2$ (but ηk_y^2 finite $\Rightarrow \gamma \neq \gamma_I$)

$\Rightarrow$

$$\cancel{\gamma^2} + \frac{k_z^2 V_A^2 \gamma}{\eta k_y^2} - \frac{g}{|L_\rho|} = 0$$

$\therefore$

$$\gamma \cong \frac{\gamma_I^2 \eta k_y^2}{k_z^2 V_A^2}$$

$$\cong \gamma_I^2 \eta k_y^2 L^2 / V_A^2$$

$L \Rightarrow$ system length

Note:

- γ independent k_x, and $\sim k_y^2$ (but $\gamma \sim \gamma_I$ as $\eta k_y^2 \to \infty$).

- rapid variation is in $y \Rightarrow$ modes slice, so alternate layers parallel to B_0 move up and down. Growth bigger for smaller slice.

- growth depends on ηL^2 (system length appears) $\Rightarrow$ reflects fact that energy into bent field lines

- $$\gamma \frac{\delta n}{n} = -\frac{\hat{V}_r}{L_p} = \frac{\gamma \hat{\xi}}{L_p}$$

$$\hat{\xi} = \frac{\delta n}{n} L_p$$ $\Rightarrow$ vertical ($\hat{x}$) displacement related to $\delta n/n$

Now, approximate equilibrium if:

$$-\delta x = \frac{\gamma_I^2 L_p}{k_z^2 V_A^2} \Rightarrow$$ i.e. displacement of B_0 for local eqbrm

but bending $B_0 \Rightarrow \tilde{B}_\perp$. When dissipated, $(\eta!)$ growth can proceed $\Rightarrow$

$$\frac{1}{x}\frac{dx}{dt} = \left(\frac{\gamma_I^2}{k_z^2 V_A^2}\right)\underbrace{\eta k_y^2}_{\text{diss. rate}} \Rightarrow$$ basic growth scaling!!

i.e. resistive diffusion destroys 'line-bent' equilibrium!

○ Finite Shear, no line-tying

$$\nabla_\perp^2 \hat{\phi} + \frac{V_A^2}{\gamma\eta}\nabla_\parallel^2 \hat{\phi} + \frac{g}{k_0 l \gamma^2} k_y^2 \hat{\phi} = 0$$

in twisting coordinates: $\begin{cases} \chi = y - \frac{x}{L_s} z \\ \xi = x \\ \varphi = z \end{cases}$

$$\nabla_\parallel \to \frac{\partial}{\partial \varphi}$$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial}{\partial y} = \frac{d\xi}{dy}\frac{\partial}{\partial y} + \frac{d\chi}{dy}\frac{\partial}{\partial \chi} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \chi}$$

$$\frac{\partial}{\partial x} = \frac{d\xi}{dx}\frac{\partial}{\partial \xi} + \frac{d\chi}{dx}\frac{\partial}{\partial \chi} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

$$= \frac{\partial}{\partial \xi} - \frac{z}{L_s}\frac{\partial}{\partial \chi}$$

$$= \frac{\partial}{\partial \xi} - \frac{\varphi}{L_s}\frac{\partial}{\partial \chi}$$, in twisting coords.

$\Rightarrow$

$$\left[\left(\frac{\partial}{\partial \xi} - \frac{\varphi}{L_s}\frac{\partial}{\partial \chi}\right)^2 + \frac{\partial^2}{\partial \chi^2}\right]\tilde{\phi} + \frac{V_A^2}{\gamma\eta}\frac{\partial^2 \tilde{\phi}}{\partial \varphi^2}$$

$$- \frac{g}{|L_e|\gamma}\frac{\partial^2}{\partial \chi^2}\tilde{\phi} = 0$$

$$\tilde{\phi} = e^{ik_x\xi} e^{ik_y\chi} \tilde{\phi}(\varphi)$$
$$= e^{ik_x x} e^{ik_y(y - \frac{x}{L_s}z)} \tilde{\phi}(z)$$

$\hookrightarrow$ {eigenfunction describes length along field line

$\Rightarrow$

$$-\left[\left(k_x - \frac{k_y \varphi}{L_s}\right)^2 + k_y^2\right]\tilde{\phi} + \frac{V_A^2}{\gamma\eta}\frac{\partial^2 \tilde{\phi}}{\partial \varphi^2} + \frac{k_y^2 g}{|L_e|\gamma^2}\tilde{\phi} = 0$$

$$\frac{\partial^2 \tilde{\phi}}{\partial \varphi^2} - \frac{\gamma\eta k_y^2}{V_A^2}\left[\left(\frac{k_x}{k_y} - \frac{\varphi}{L_s}\right)^2 + 1\right]\tilde{\phi} + \frac{k_y^2 g\,\gamma\eta}{|L_e| V_A^2 \gamma^2}\tilde{\phi} = 0$$

– if $k_x \ll k_y \Rightarrow$

$$\frac{\partial^2 \tilde{\phi}}{\partial \varphi^2} - \frac{\gamma \eta k_y^2}{V_A^2}\left[\frac{\varphi^2}{L_S^2} + 1\right]\tilde{\phi} + \frac{\eta k_y^2 \gamma_I^2 \tilde{\phi}}{\gamma V_A^2} = 0$$

$$\tilde{\phi}(\varphi) = e^{-\alpha \varphi^2/2}$$

$$\alpha^2 \varphi^2 - \alpha - \frac{\gamma \eta k_y^2}{V_A^2 L_S^2}\varphi^2 - \cancel{\frac{\gamma \eta k_y^2}{V_A^2}} + \frac{\eta k_y^2 \gamma_I^2}{\gamma V_A^2} = 0$$

$(\because \varphi^2 > L_S^2)$

$$\alpha = \left(\frac{\gamma \eta k_y^2}{V_A^2 L_S^2}\right)^{1/2} \qquad \left(\frac{\gamma \eta k_y^2}{V_A^2 L_S^2}\right)^{1/2} = \frac{\eta k_y^2 \gamma_I^2}{\gamma V_A^2}$$

$$\Rightarrow \begin{cases} \gamma \tau_A \sim S^{-1/3}, \quad \gamma \sim \eta^{1/3} \\ \sqrt{\alpha} \sim 1/L_{eff} \sim \eta^{1/3} \quad \Rightarrow L_{eff} \sim \eta^{-1/3} \end{cases}$$

Note:

– as all dynamics independent of k_x can take $k_x a \sim 1$ and

- $\phi = F(x)\,\hat{\phi}(z)\, e^{ik_y(y - \frac{x}{L_s} z)}$

  $\hat{\phi}(z)$: eigenfunction $L \sim \eta^{-1/3}$

  $F(x)$: arbitrary envelope in x, slowly varying in comparison k_y

- "modes" are finite <u>length</u> rolls, <u>locally</u> aligned with shear:

  <u>i.e.</u> pattern ϕ constant along $y - \frac{xz}{L_s} \equiv$ const.

- "modes" have finite length in z (allow twist to align with shear).

- $\gamma \sim \eta^{1/3}$ but $k_x a \sim 1$ $\Rightarrow$ large transport!

- can derive from local analysis with $L \sim \eta^{-1/3}$ <u>i.e.</u>; energetics determine length (must keep rotational energy <u>finite</u>)

- "modes" are really <u>wave-packets</u> of localized g-modes (quasi-modes)

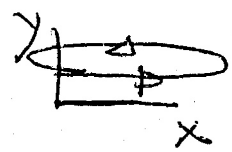
  i.e. $k_\parallel = \frac{k_y x}{L_s}$ $\quad (e^{ik_\parallel L})$

  $\frac{\partial}{\partial x} = \frac{k_y}{L_s} \frac{\partial}{\partial k_\parallel}$

  $\Rightarrow \quad \partial/\partial L \rightarrow \frac{k_y}{L_s} L$ — "radial and "along field" spaces reciprocal with length of envelope !

key point is insensitivity to x variation (~ const profile) allows superposition, s/t:

- $k_x \ll k_y$



- natural for growth (large step)

⇒ dynamically favored

i.e. localization in $\hat{x}$ is consequence Fourier analysis, not physical nature of disturbance !!

- in nonlinear regime: (?)

  - packets may fall apart

  - $\underline{k}\cdot\underline{k}'\times\hat{z} \to \frac{k_y k_y'}{L_s} z\,(x-x) = 0$

    coupling ?

  - sub-scale instability